On discrete Borell-Brascamp-Lieb type inequalities

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Conference on Convex, Discrete and Integral Geometry

Universität Jena

September 16th, 2019

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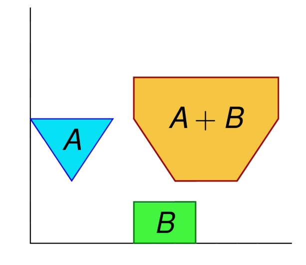

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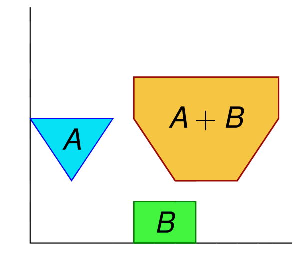

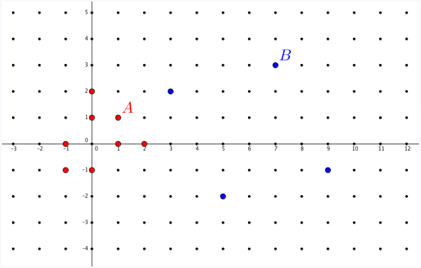

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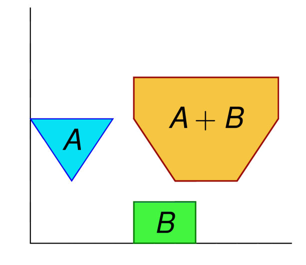

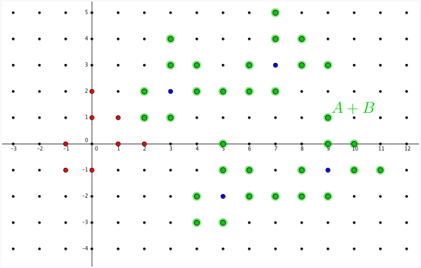

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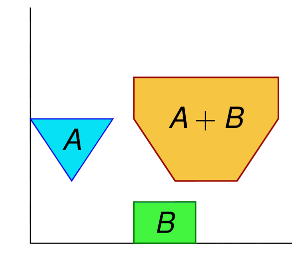

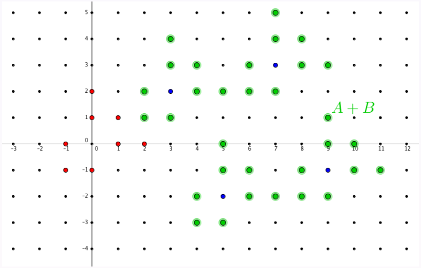

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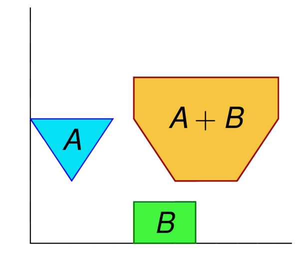

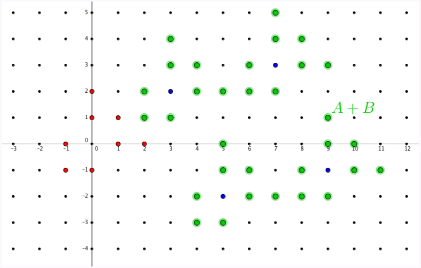

- $\mathrm{vol}(K)$ = volume (Lebesgue measure) of $K \subset \mathbb{R}^n$.
- $|A|$ = cardinality of $A \subset \mathbb{R}^n$ finite.
- $\mathrm{G}_n(M) = |M \cap \mathbb{Z}^n|$ = lattice point enumerator of $M \subset \mathbb{R}^n$ bounded.

The Brunn-Minkowski inequality

Relating the volume with the Minkowski addition of compact sets (not necessarily convex), one is led to the famous Brunn-Minkowski inequality:

The Brunn-Minkowski inequality (additive version)

Let $K, L \subset \mathbb{R}^n$ be (non-empty) compact sets. Then

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From the homogeneity of the volume, this inequality is equivalent to the following one:

The Brunn-Minkowski inequality ($(1/n)$-concave version)

Let $K, L \subset \mathbb{R}^n$ be (non-empty) compact sets and let $\lambda \in (0,1)$. Then

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}.$$

Notation: p-sums and p-means

- S_p denotes the *p-sum* of two non-negative numbers a, b, for $p \neq 0$:

$$S_p(a,b) = \begin{cases} \left(a^p + b^p\right)^{1/p}, & \text{if } p \neq 0, \pm\infty, \\ \max\{a,b\} & \text{if } p = \infty, \\ \min\{a,b\} & \text{if } p = -\infty; \end{cases}$$

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- M_p denotes the *p-mean* of two non-negative numbers a, b:

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The functional version of the Brunn-Minkowski inequality

The Borell-Brascamp-Lieb inequality

Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be measurable functions.

- (p-sums) If f, g, h are such that $h(x+y) \geq S_p(f(x), g(y))$, $p \neq 0$, for all $x, y \in \mathbb{R}^n$ then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq S_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right).$$

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By applying this result to the functions $f = \chi_{_A}$, $g = \chi_{_B}$ and $h = \chi_{_{A+B}}$ (respectively $h = \chi_{_{(1-\lambda)A+\lambda B}}$), and $p = \infty$, the above versions of the Brunn-Minkowski inequality are recovered.

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- We consider either **finite subsets** $A, B \subset \mathbb{Z}^n$ **of integer points**, and our measure will be the **cardinality**,

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Does any discrete Brunn-Minkowski inequality exist in the classical form? Namely, is it true that

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n} \ ?$$

And what about

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n} \ ?$$

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NO!

For $A = \{0\}$ and any $B \subset \mathbb{Z}^n$ finite,

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For $K = \{0\}$ and $L = [0, m]^n$, with $m \in \mathbb{N}$ odd, and $\lambda = 1/2$,

$$\frac{m+1}{2} = \mathrm{G}_n\big((1-\lambda)K + \lambda L\big)^{1/n} < (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n} = \frac{m+2}{2}.$$

Some bounds for $|A+B|$

A linear discrete Brunn-Minkowski inequality

Let $A, B \subset \mathbb{R}^n$ be finite sets. Then

$$|A+B| \geq |A| + |B| - 1.$$

Ruzsa, 1994

If $A, B \subset \mathbb{R}^n$ are finite sets with $|B| \leq |A|$ and $\dim(A+B) = n$, then

$$|A+B| \geq |A| + n|B| - \frac{n(n+1)}{2},$$

and there is no improvement of it linear in $|A|$.

A discrete Brunn-Minkowski inequality by Gardner-Gronchi

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If $A, B \subset \mathbb{Z}^n$ are finite sets with $\dim B = n$, then

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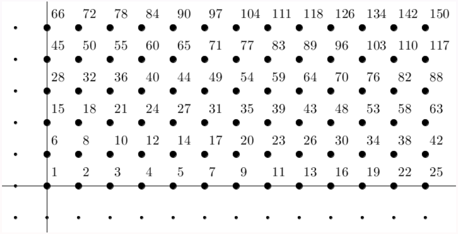

$|B| = 6$

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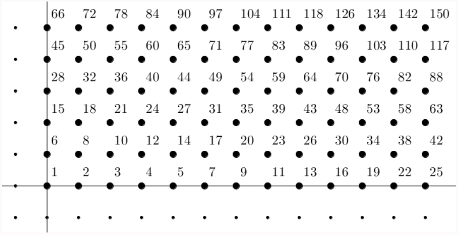
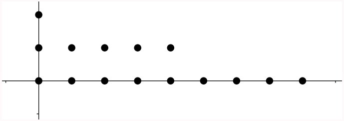


$|B| = 6$ and $|A| = 15$



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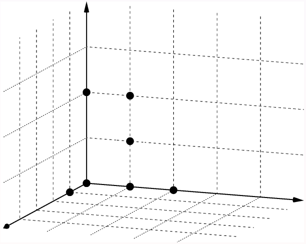

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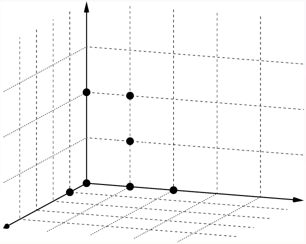

- Let $m_0 \in \pi_k(F)$ be such that

$$\left|F(m_0)\right| = \max_m \left|F(m)\right|.$$

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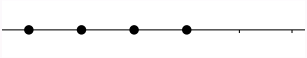

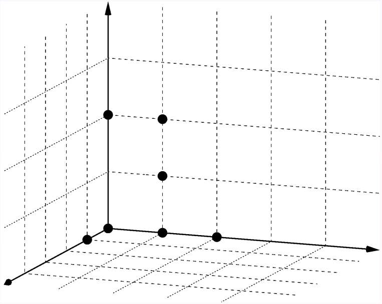

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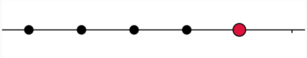

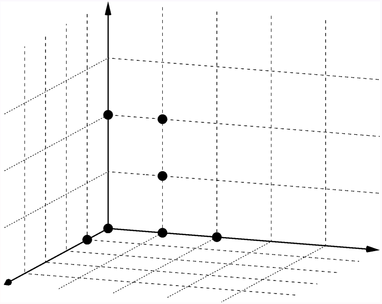

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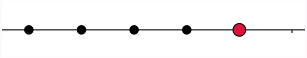

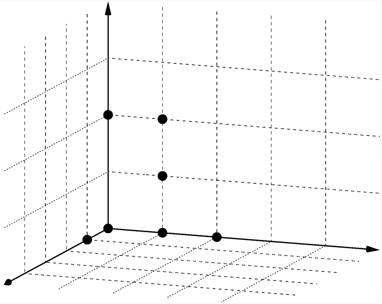

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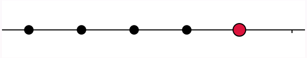
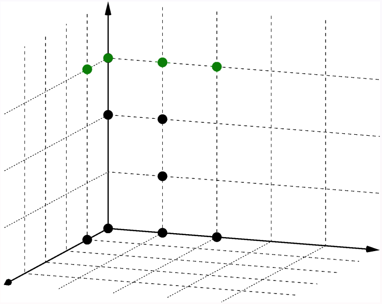
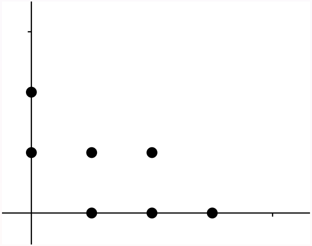




Extending one of the involved sets

- First step: $A_1 = \sigma_n(A)$
- Second step: $A_2 = \displaystyle\bigcup_{m \in \pi_n(A_1)} \Big(\sigma_{n-1}\big(A_1(m)\big) \times \{m\}\Big)$.
- k-th step: $A_k = \displaystyle\bigcup_{m \in \pi_{n,\dots,n-k+2}(A_{k-1})} \Big(\sigma_{n-k+1}\big(A_{k-1}(m)\big) \times \{m\}\Big)$.
- We define $\bar{A} = A_n$.



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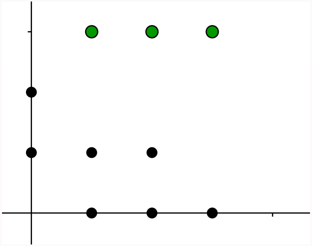

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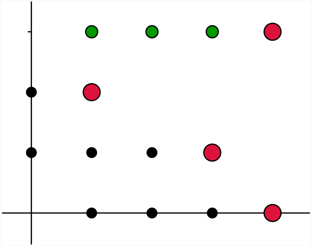

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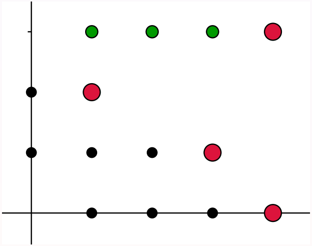

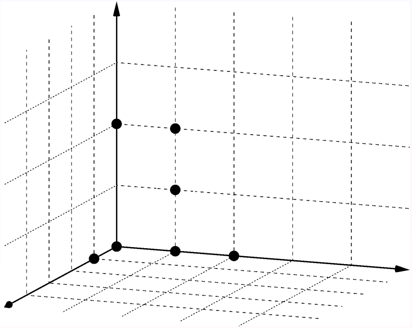

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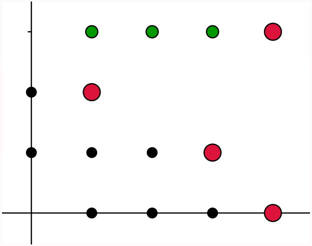

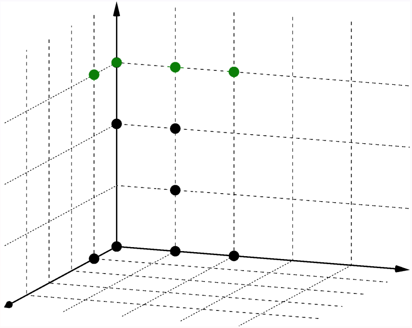

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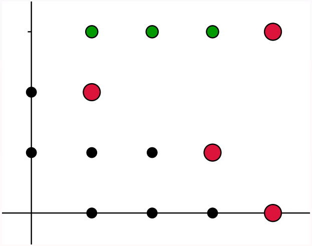

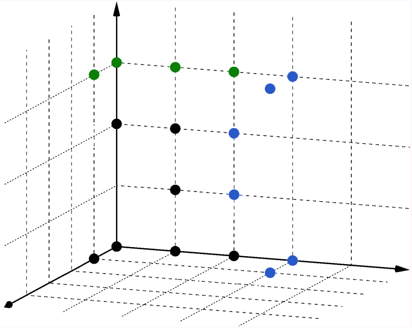

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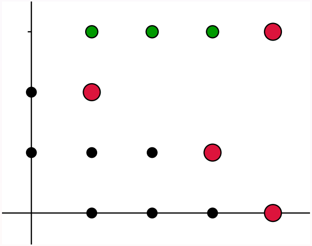

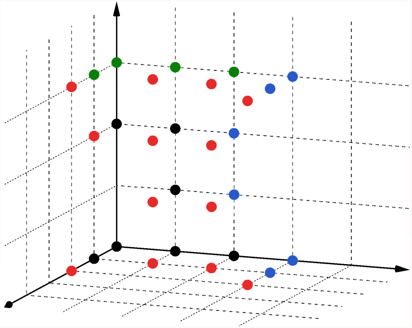

A new discrete Brunn-Minkowski inequality

Theorem (Hernández Cifre, Iglesias, Y. N. (2018))

Let A, B be finite subsets of $\mathbb{Z}^n$, $A, B \neq \emptyset$. Then

$$\left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

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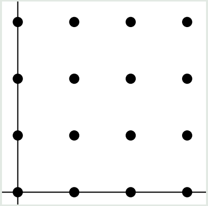
Lattice cubes

For $m \in \mathbb{N}$, $\mathcal{C}_m = [0, m]^n \cap \mathbb{Z}^n$.



Equality holds if $A = \mathcal{C}_{m_1}, B = \mathcal{C}_{m_2}$ are lattice cubes:

- $\bar{A} = \mathcal{C}_{m_1+1} \Longrightarrow \bar{A} + B = \mathcal{C}_{m_1+m_2+1}$.
- $\left|\bar{A} + B\right| = (m_1 + m_2 + 2)^n$
  $= \left(|A|^{1/n} + |B|^{1/n}\right)^n.$

On a discrete BBL inequality

Regarding a discrete version of the Borell-Brascamp-Lieb inequality (for p-sums) when dealing with finite sets we have:

Theorem (Iglesias, Y. N. (2019))

Let $A, B \subset \mathbb{Z}^n$ be finite sets. Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be functions such that $h(x+y) \geq S_p(f(x), g(y))$ for all $x \in A$, $y \in B$. Then

$$\sum_{z \in A+B} h(z) \geq S_{\frac{p}{np+1}}\left(\sum_{x \in \mathfrak{r}_f(A)} f(x), \sum_{y \in B} g(y)\right).$$

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- When $f = g \equiv 1$ and $p = \infty$, the above discrete Brunn-Minkowski inequality is recovered.

On discrete BM and BBL inequalities for $\mathrm{G}_n(\cdot)$

Theorem (Iglesias, Y. N., Zvavitch (2019+))

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets and let $\lambda \in (0,1)$. Let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be functions such that $h((1-\lambda)x + \lambda y) \geq M_p(f(x), g(y), \lambda)$ for all $x \in K$, $y \in L$. Then

$$\sum_{z \in M \cap \mathbb{Z}^n} h^*(z) \geq M_{\frac{p}{np+1}} \left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y), \lambda \right),$$

where $M = (1-\lambda)K + \lambda L + (-1,1)^n$.

Here, for $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, ϕ^* denotes the function $\phi^* : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by $\phi^*(z) = \sup_{u \in (-1,1)^n} \phi(z + u)$ for all $z \in \mathbb{R}^n$.

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Let $K, L \subset \mathbb{R}^n$ be bounded sets with $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$ and let $\lambda \in (0,1)$. Then

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

The inequality is sharp.

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Not possible to obtain any of the above discrete BM inequalities from the classical ones by replacing the points by suitable compact sets.

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On discrete Borell-Brascamp-Lieb type inequalities

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September 16th, 2019