Refined concentration of measure derived from convex infimum convolution inequalities

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Concentration for convex functions

- $G \sim \mathcal{N}(0, \mathrm{I}_n)$
- $f : \mathbb{R}^n \to \mathbb{R}$ – L-Lipschitz

$$\implies \mathbb{P}(|f(G) - \operatorname{Med} f(G)| \geq t) \leq 2 \exp\left(-\frac{t^2}{2L^2}\right)$$

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Theorem (Talagrand, '90s)

- $X_1, \ldots, X_n$ – independent r.v., $|X_i| \leq 1$
- $X := (X_1, \ldots, X_n)$,
- $f : \mathbb{R}^n \to \mathbb{R}$ – convex, L-Lipschitz

$$\implies \mathbb{P}(|f(X) - \operatorname{Med} f(X)| \geq t) \leq 4 \exp\Big(-\frac{t^2}{16L^2}\Big)$$

A Gaussian small deviation inequality for convex functions

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Theorem (Paouris–Valettas, 2018)

- $G \sim \mathcal{N}(0, \mathrm{I}_n)$
- $f : \mathbb{R}^n \to \mathbb{R}$ – convex with $\mathbb{E}f(G)^2 < \infty$

$$\implies \mathbb{P}\big(f(G) \leq \mathrm{Med}\, f(G) - t\big) \leq \frac{1}{2}\exp\Big(-\frac{ct^2}{\mathrm{Var}\, f(G)}\Big)$$

Convex infimum convolution inequality

- $\theta\colon \mathbb{R}^n \to [0,\infty)$ – convex, symm., $\theta(x) = 0 \Leftrightarrow x = 0$,

$$B_\theta(r) := \{x \in \mathbb{R}^n : \theta(x) < r\}$$

- $f : \mathbb{R}^n \to \mathbb{R}$ – bdd from below

$$f \Box \theta(x) := \inf_{y \in \mathbb{R}^n} \{f(y) + \theta(x - y)\}$$

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Theorem (Maurey, 1991)

Assume $\mathbb{E}e^{f\Box\theta(X)}\mathbb{E}e^{-f(X)} \leq 1$ for all convex $f\colon \mathbb{R}^n \to \mathbb{R}$ bdd from below. Then for any convex $A \subseteq \mathbb{R}^n$ and $r > 0$,

$$\mathbb{P}(X \notin A + B_\theta(r))\mathbb{P}(X \in A) \leq e^{-r}.$$

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Proof. $f = \infty 1_{\{x \notin \mathrm{cl}\, A\}}$; $f \square \theta(x) < r \Leftrightarrow \exists y \in A$ s.t. $\theta(x - y) < r$

$$e^r \mathbb{P}(X \notin A + B_\theta(r)) \mathbb{P}(X \in A) \leq \mathbb{E} e^{f \square \theta(X)} \mathbb{E} e^{-f(X)} \leq 1.$$

$$|x|_{\frac{1}{p}\theta} := \inf\{a > 0\colon \theta(x/a) \le p\}, \quad |x|_{\theta,p} := \sup\Big\{\sum_i x_i y_i \colon \theta(y) \le p\Big\}$$

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Apply to

$$A = \{y \in \mathbb{R}^n\colon f(y) \le \operatorname{Med} f(X)\},$$
$$\widetilde{A} = \{y \in \mathbb{R}^n\colon f(y) < \operatorname{Med} f(X) - \sup_{x \in \mathbb{R}^n} |\nabla f(x)|_{\theta,p}\}.$$

Corollary

For any smooth convex Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$ and $p \ge 0$,

$$\mathbb{P}\Big(|f(X) - \operatorname{Med} f(X)| > \sup_{x \in \mathbb{R}^n} |\nabla f(x)|_{\theta,p}\Big) \le 4e^{-p}.$$

Coming to the topic of the talk...

- X – r.v. in $\mathbb{R}^n$, s.t. for all smooth convex Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$

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Theorem (Adamczak–St., 2019)

For any smooth convex $f\colon \mathbb{R}^n \to \mathbb{R}$ and any $p \ge 1$,

(i) $$\Big\| \frac{(f(X) - \operatorname{Med} f(X))_+}{|\nabla f(X)|_{\theta,p}} \Big\|_p \le 3^{1/p}.$$

Let $\mathbb{P}(|\nabla f(X)|_{\theta,p} \le M_p) \ge 3/4$. Then, for any $p > 0$,

(ii) $$\mathbb{P}\Big(f(X) < \operatorname{Med} f(X) - 4M_p\Big) \le 4e^{-p},$$

(ii') $$\mathbb{P}\Big(f(X) < \operatorname{Med} f(X) - 16\mathbb{E}|\nabla f(X)|_{\theta,p}\Big) \le 4e^{-p}.$$

A specific example

- $r \geq 1, \quad \theta(x) = c|x|^r, \quad |x|_{\theta,p} = c^{-1/r}p^{1/r}|x|$
- X – r.v. in $\mathbb{R}^n$, s.t. for all smooth convex 1-Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$

$$\mathbb{P}(|f(X) - \operatorname{Med} f(X)| > t) \leq 4\exp(-ct^r). \qquad (*)$$

(i) $$\Big\| \frac{(f(X) - \operatorname{Med} f(X))_+}{|\nabla f(X)|} \Big\|_p \leq 3^{1/p} c^{-1/r} p^{1/r}$$

$$\overset{Cheb.}{\Longrightarrow} \mathbb{P}\Big(\frac{f(X) - \operatorname{Med} f(X)}{|\nabla f(X)|} \geq t \Big) \leq e \exp\Big(- \frac{ct^r}{(3e)^r} \Big)$$

(ii') $$\mathbb{P}\Big(f(X) \leq \operatorname{Med} f(X) - t \Big) \leq 4\exp\Big(-c \frac{t^r}{16^r (\mathbb{E}|\nabla f(X)|)^r} \Big)$$