Bounds for expectations of k-maxima of log-concave vectors

Marta Strzelecka

(Based on joint work with Rafał Latała)

University of Warsaw

Convex, Discrete and Integral Geometry, Jena

September 16, 2019

By $k\text{-}\max_i x_i$ we denote the k-th maximal value of $(x_i)_{i\leq n}$.

By $k\text{-}\max_i x_i$ we denote the k-th maximal value of $(x_i)_{i\leq n}$.
By $k\text{-}\min_i x_i$ we denote the k-th minimal value of $(x_i)_{i\leq n}$.

By $k\text{-}\max_i x_i$ we denote the k-th maximal value of $(x_i)_{i\leq n}$.
By $k\text{-}\min_i x_i$ we denote the k-th minimal value of $(x_i)_{i\leq n}$.
Aim: X – random vector. Find upper and lower estimates of

$$\mathbb{E}k\text{-}\max_i |X_i| \qquad \text{and} \qquad \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| = \mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i|$$

by the same quantity (up to universal constants).

By $k\text{-}\max_i x_i$ we denote the k-th maximal value of $(x_i)_{i\le n}$.
By $k\text{-}\min_i x_i$ we denote the k-th minimal value of $(x_i)_{i\le n}$.
Aim: X – random vector. Find upper and lower estimates of

$$\mathbb{E} k\text{-}\max_i |X_i| \qquad \text{and} \qquad \mathbb{E} \max_{|I|=k} \sum_{i\in I} |X_i| = \mathbb{E} \sum_{l=1}^{k} l\text{-}\max_i |X_i|$$

by the same quantity (up to universal constants).
To estimate

$$\mathbb{E} \max_{|I|=k} \sum_{i\in I} |X_i|$$

we will need the parameter

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i|\ge t\}} \le k \right\}.$$

$$t(k, X) := \inf \left\{ t > 0 \colon \frac{1}{t} \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} \leq k \right\}.$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

For $t > t(k,X)$,

$$\max_{|I|=k}\sum_{i\in I}|X_i| \leq \max_{|I|=k}\sum_{i\in I}|X_i|\mathbf{1}_{\{|X_i|<t\}} + \max_{|I|=k}\sum_{i\in I}|X_i|\mathbf{1}_{\{|X_i|\geq t\}}$$
$$\leq tk + \sum_{i=1}^{n}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq 2tk,$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

For $t > t(k,X)$,

$$\begin{aligned}\max_{|I|=k}\sum_{i\in I}|X_i| \leq \max_{|I|=k}\sum_{i\in I}|X_i|\mathbf{1}_{\{|X_i|<t\}} + \max_{|I|=k}\sum_{i\in I}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \\ \leq tk + \sum_{i=1}^{n}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq 2tk,\end{aligned}$$

so

$$\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

$$t(k,X) := \inf \left\{ t > 0 \colon \frac{1}{t} \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} \leq k \right\}.$$

For $t > t(k,X)$,

$$\begin{aligned} \max_{|I|=k} \sum_{i \in I} |X_i| &\leq \max_{|I|=k} \sum_{i \in I} |X_i| \mathbf{1}_{\{|X_i| < t\}} + \max_{|I|=k} \sum_{i \in I} |X_i| \mathbf{1}_{\{|X_i| \geq t\}} \\ &\leq tk + \sum_{i=1}^{n} |X_i| \mathbf{1}_{\{|X_i| \geq t\}} \leq 2tk, \end{aligned}$$

so

$$\max_{|I|=k} \sum_{i \in I} |X_i| \leq 2kt(k,X).$$

Question: For which random vectors X,

$$\max_{|I|=k} \sum_{i \in I} |X_i| \geq ckt(k,X)?$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

$$t(k, X) := \inf \left\{ t > 0 \colon \frac{1}{t} \sum_{i=1}^{n} \mathbb{E}|X_i| \mathbf{1}_{\{|X_i| \geq t\}} \leq k \right\}.$$

Let X satisfy

$$\mathbb{P}(|X_i| \geq s, |X_j| \geq t) \leq \alpha \mathbb{P}(|X_i| \geq s) \mathbb{P}(|X_j| \geq t) \tag{1}$$

for all $i \neq j$ and all $s, t > 0$. $(\alpha \geq 1)$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Let X satisfy

$$\mathbb{P}(|X_i|\geq s, |X_j|\geq t) \leq \alpha \mathbb{P}(|X_i|\geq s)\mathbb{P}(|X_j|\geq t) \tag{1}$$

for all $i\neq j$ and all $s,t>0$. $(\alpha\geq 1)$
Note that random vectors with independent coordinates satisfy (1).

$$t(k,X) := \inf\left\{t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k\right\}.$$

Let X satisfy

$$\mathbb{P}(|X_i|\geq s, |X_j|\geq t) \leq \alpha\mathbb{P}(|X_i|\geq s)\mathbb{P}(|X_j|\geq t) \tag{1}$$

for all $i\neq j$ and all $s,t>0$. $(\alpha\geq 1)$

Theorem (Latała-St. '19)

Then there exists a constant $c(\alpha)>0$ which depends only on α such that for any $1\leq k\leq n$,

$$c(\alpha)kt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

We may take $c(\alpha) = (288(5+4\alpha)(1+2\alpha))^{-1}$.

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda} \leq \mathbb{P}(X \in \lambda A + (1-\lambda)B).$$

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda} \leq \mathbb{P}(X \in \lambda A + (1-\lambda)B).$$

Roughly speaking, X is log-concave iff it has log-concave density.

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda} \leq \mathbb{P}(X \in \lambda A + (1-\lambda)B).$$

Roughly speaking, X is log-concave iff it has log-concave density.

Log-concavity = vectors uniformly distributed on convex bodies
+ affine transformations + weak limits.

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda} \leq \mathbb{P}(X \in \lambda A + (1-\lambda)B).$$

Roughly speaking, X is log-concave iff it has log-concave density.

Log-concavity $=$ vectors uniformly distributed on convex bodies
$+$ affine transformations $+$ weak limits.

Log-concave vectors with diagonal covariance matrices behave in many aspects like vectors with independent coordinates. This is true also in our case:

Definition

We say that a random vector X in $\mathbb{R}^n$ is log-concave if for any compact subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$ we have

$$\mathbb{P}(X \in A)^{\lambda}\mathbb{P}(X \in B)^{1-\lambda} \leq \mathbb{P}(X \in \lambda A + (1-\lambda)B).$$

Roughly speaking, X is log-concave iff it has log-concave density.
Log-concavity = vectors uniformly distributed on convex bodies
+ affine transformations + weak limits.
Log-concave vectors with diagonal covariance matrices behave in many aspects like vectors with independent coordinates.

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates** *(i.e.* $\mathrm{Cov}(X_i, X_j) = 0$ *for* $i \neq j$*). Then for any* $1 \leq k \leq n$,

$$ckt(k, X) \leq \mathbb{E} \max_{|I|=k} \sum_{i \in I} |X_i| \leq 2kt(k, X).$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates**. *Then*

$$ckt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates**. *Then*

$$ckt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

Example 1. Let $X = (\varepsilon_1 g, \varepsilon_2 g, \ldots, \varepsilon_n g)$, where $\varepsilon_1, \ldots, \varepsilon_n, g$ are independent, $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$ and g has the normal $\mathcal{N}(0,1)$ distribution.

$$t(k,X) := \inf\left\{t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k\right\}.$$

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates***. Then*

$$ckt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

Example 1. Let $X=(\varepsilon_1 g, \varepsilon_2 g, \ldots, \varepsilon_n g)$, where $\varepsilon_1,\ldots,\varepsilon_n, g$ are independent, $\mathbb{P}(\varepsilon_i=\pm 1)=1/2$ and g has the normal $\mathcal{N}(0,1)$ distribution. Then $\operatorname{Cov} X = \mathrm{Id}$ and it is not hard to check that $\mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| = k\sqrt{2/\pi}$ and $t(k,X)\sim \ln^{1/2}(n/k)$ if $k\leq n/2$.

$$t(k,X) := \inf\left\{t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k\right\}.$$

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates***. Then*

$$ckt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

Example 1. Let $X = (\varepsilon_1 g, \varepsilon_2 g, \ldots, \varepsilon_n g)$, where $\varepsilon_1, \ldots, \varepsilon_n, g$ are independent, $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$ and g has the normal $\mathcal{N}(0,1)$ distribution. Then $\operatorname{Cov} X = \mathrm{Id}$ and it is not hard to check that $\mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| = k\sqrt{2/\pi}$ and $t(k,X) \sim \ln^{1/2}(n/k)$ if $k \leq n/2$.

Example 2. Let $X = (g, \ldots, g)$, where $g \sim \mathcal{N}(0,1)$.

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Theorem (Latała-St. '19)

Let X be a **log-concave** *random vector with* **uncorrelated coordinates**. *Then*

$$ckt(k,X) \leq \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X).$$

Example 1. Let $X = (\varepsilon_1 g, \varepsilon_2 g, \ldots, \varepsilon_n g)$, where $\varepsilon_1, \ldots, \varepsilon_n, g$ are independent, $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$ and g has the normal $\mathcal{N}(0,1)$ distribution. Then $\operatorname{Cov} X = \mathrm{Id}$ and it is not hard to check that $\mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| = k\sqrt{2/\pi}$ and $t(k,X) \sim \ln^{1/2}(n/k)$ if $k \leq n/2$.

Example 2. Let $X = (g, \ldots, g)$, where $g \sim \mathcal{N}(0,1)$. Then $\mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| = k\sqrt{2/\pi}$ and $t(k,X) \sim \ln^{1/2}(n/k)$ if $k \leq n/2$.

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Recall that

$$\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| = \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X),$$

so

$$\mathbb{E}k\text{-}\max_i |X_i| \leq \frac{1}{k}\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| \leq 2t(k,X).$$

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n} \mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Recall that

$$\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| = \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X),$$

so

$$\mathbb{E}k\text{-}\max_i |X_i| \leq \frac{1}{k}\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| \leq 2t(k,X).$$

But we cannot hope that a reverse inequality holds even for nice vectors X.

$$t(k,X) := \inf\left\{ t > 0\colon \frac{1}{t}\sum_{i=1}^{n} \mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Recall that

$$\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| = \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X),$$

so

$$\mathbb{E}k\text{-}\max_i |X_i| \leq \frac{1}{k}\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| \leq 2t(k,X).$$

But we cannot hope that a reverse inequality holds even for nice vectors X. If a half of coordinates of X is equal to 0 a.s., then $\mathbb{E}\frac{n}{2}\text{-}\max_i |X_i| = 0$, but $t(k,X) > 0$ for all k.

$$t(k,X) := \inf\left\{ t>0\colon \frac{1}{t}\sum_{i=1}^{n}\mathbb{E}|X_i|\mathbf{1}_{\{|X_i|\geq t\}} \leq k \right\}.$$

Recall that

$$\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| = \mathbb{E}\max_{|I|=k}\sum_{i\in I}|X_i| \leq 2kt(k,X),$$

so

$$\mathbb{E}k\text{-}\max_i |X_i| \leq \frac{1}{k}\mathbb{E}\sum_{l=1}^{k} l\text{-}\max_i |X_i| \leq 2t(k,X).$$

But we cannot hope that a reverse inequality holds even for nice vectors X. If a half of coordinates of X is equal to 0 a.s., then $\mathbb{E}\frac{n}{2}\text{-}\max_i |X_i| = 0$, but $t(k,X)>0$ for all k.

We introduce a parameter which is more sensitive for rescaling a single coordinate.

$$t^*(p,X) := \inf\left\{t>0\colon \ \sum_{i=1}^{n}\mathbb{P}(|X_i|\geq t)\leq p\right\} \quad \text{for } 0<p<n.$$

$$t^*(p,X) := \inf\left\{t>0\colon \ \sum_{i=1}^n \mathbb{P}(|X_i|\geq t)\leq p\right\} \quad \text{for } 0<p<n.$$

Theorem (Latała-St. '19)

Let X be a mean zero **log-concave** *n-dimensional random vector with* **uncorrelated coordinates** *and $1\leq k\leq n$. Then*

$$\mathbb{E}k\text{-}\max_{i\leq n}|X_i| \geq \frac{1}{2}\mathrm{Med}\left(k\text{-}\max_{i\leq n}|X_i|\right) \geq ct^*\left(k-\frac{1}{2}, X\right).$$

$$t^*(p,X) := \inf\left\{t>0\colon \ \sum_{i=1}^n \mathbb{P}(|X_i|\geq t)\leq p\right\} \quad \text{for } 0<p<n.$$

Theorem (Latała-St. '19)

Let X be a mean zero **log-concave** *n-dimensional random vector with* **uncorrelated coordinates** *and $1 \leq k \leq n$. Then*

$$\mathbb{E}k\text{-}\max_{i\leq n}|X_i| \geq \frac{1}{2}\mathrm{Med}\left(k\text{-}\max_{i\leq n}|X_i|\right) \geq ct^*\left(k-\frac{1}{2},X\right).$$

The proof uses a lower bound for sums of k biggest $|X_i|$.

$$t^*(p, X) := \inf\left\{ t > 0 \colon \ \sum_{i=1}^{n} \mathbb{P}(|X_i| \geq t) \leq p \right\} \quad \text{for } 0 < p < n.$$

Theorem (Latała-St. '19)

Let X be a mean zero **log-concave** *n-dimensional random vector with* **uncorrelated coordinates** *and $1 \leq k \leq n$. Then*

$$\mathbb{E} k\text{-}\max_{i \leq n} |X_i| \geq \frac{1}{2} \mathrm{Med}\Big(k\text{-}\max_{i \leq n} |X_i| \Big) \geq c t^* \left(k - \frac{1}{2}, X \right).$$

The proof uses a lower bound for sums of k biggest $|X_i|$. Although the parameter t^* is more adequate than t to estimate $\mathbb{E} k\text{-}\max_{i \leq n} |X_i|$, not only the lower bound, but also the upper bound is non-trivial. The upper bound turns out to be even more challenging (for us):

$$t^*(p,X) := \inf\Big\{t>0\colon \ \sum_{i=1}^{n} \mathbb{P}(|X_i|\geq t) \leq p\Big\} \quad \text{for } 0<p<n.$$

Theorem (Latała-St. '19)

Let X be a mean zero **log-concave** *n-dimensional random vector with* **uncorrelated coordinates** *and $1 \leq k \leq n$. Then*

$$\mathbb{P}\left(k\text{-}\max_{i\leq n}|X_i| \geq Ct^*\left(k-\frac{1}{2},X\right)\right) \leq 1-c \tag{2}$$

and

$$\mathbb{E}k\text{-}\max_{i\leq n}|X_i| \leq Ct^*\left(k-\frac{1}{2}k^{5/6},X\right). \tag{3}$$

Moreover, if X is additionally **unconditional** *then*

$$\mathbb{E}k\text{-}\max_{i\leq n}|X_i| \leq Ct^*\left(k-\frac{1}{2},X\right).$$

The unconditionality assumption plays a crucial role in the proof of the next lemma, which allows to derive (3) from (2).

Lemma

Let X be an unconditional log-concave n-dimensional random vector. Then for any $1 \le k \le n$,

$$\mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge ut\right) \le \mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge t\right)^u \quad \textit{for } u > 1, t > 0.$$

The unconditionality assumption plays a crucial role in the proof of the next lemma, which allows to derive (3) from (2).

Lemma

Let X be an unconditional log-concave n-dimensional random vector. Then for any $1 \le k \le n$,

$$\mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge ut\right) \le \mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge t\right)^u \quad \textit{for } u > 1, t > 0.$$

Question: Is it true that for log-concave uncorrelated vectors X we have

$$\mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge ut\right) \le \mathbf{C}\mathbb{P}\left(k\text{-}\max_{i\le n}|X_i| \ge t\right)^{\mathbf{C}u} \quad \text{for } u > 1, t > 0?$$

In the case of log-concave isotropic vectors X our theorem yields:

In the case of log-concave isotropic vectors X our theorem yields: for every $1 \leq k \leq n/2$,

$$\mathbb{E} k\text{-}\max_{i \leq n} |X_i| \sim t^*(k, X) \sim t(k, X)$$

In the case of log-concave isotropic vectors X our theorem yields: for every $1 \leq k \leq n/2$,

$$\mathbb{E}k\text{-}\max_{i\leq n}|X_i| \sim t^*(k,X) \sim t(k,X)$$

and

$$c\frac{k}{n} \leq \mathbb{E}k\text{-}\min_{i\leq n}|X_i| = \mathbb{E}(n-k+1)\text{-}\max_{i\leq n}|X_i| \leq C\left(\frac{k}{n} + n^{-1/6}\right).$$

In the case of log-concave isotropic vectors X our theorem yields: for every $1 \le k \le n/2$,

$$\mathbb{E}k\text{-}\max_{i\le n} |X_i| \sim t^*(k,X) \sim t(k,X)$$

and

$$c\frac{k}{n} \le \mathbb{E}k\text{-}\min_{i\le n} |X_i| \le C\left(\frac{k}{n} + n^{-1/6}\right).$$

If X is additionally unconditional then

$$\mathbb{E}k\text{-}\min_{i\le n} |X_i| \sim \frac{k}{n}.$$