Minimal volume product of convex bodies with various symmetries

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For a convex body K in $\mathbb{R}^n$, we denote its polar w.r.t. the origin o by K°.
($K^\circ := \{Q \in \mathbb{R}^n; P \cdot Q \le 1$ for any $P \in K\}$.)

Volume product

Let K be a convex body in $\mathbb{R}^n$ and K° be its polar,

$$\mathcal{P}(K) := |K||K^\circ| := \mathrm{vol}_n(K)\ \mathrm{vol}_n(K^\circ)$$

is the volume product of K.

Fact: $\mathcal{P}$ is invariant with respect to any linear transformation $\mathcal{A} \in GL(n)$.

Mahler's conjecture [Mahler (1939)]

For any centrally symmetric (i.e. $K = -K$) convex body K in $\mathbb{R}^n$,

$$\mathcal{P}(K) = |K||K^\circ| \ge \frac{4^n}{n!}.$$

Remark

- $n = 1$ case is trivial, $n = 2$ case was shown by Mahler himself.
- Sharp upper estimate of $\mathcal{P}(K)$ is already known as the Blacshke–Santaló inequality.

[Mahler's conjecture (1939)]

$\boldsymbol{\mathcal{P}(K) \geq 4^n/n!}$ for any centrally symmetric convex body $\boldsymbol{K} \subset \mathbb{R}^{\boldsymbol{n}}$.

[Saint-Raymond, Publ. Math. Univ. Pierre et Marie Curie (1980)]

$\boldsymbol{K} \subset \mathbb{R}^{\boldsymbol{n}}$: a convex body , $\boldsymbol{K}$ is symmetric w.r.t. all coordinate plane ($\Leftrightarrow$ 1-unconditional).
Then $\boldsymbol{\mathcal{P}(K) \geq 4^n/n!}$.
(simple proof: [Meyer, Israel J. Math. (1986)])

[Reisner, Math. Scand. (1985)]

$\boldsymbol{K} \subset \mathbb{R}^{\boldsymbol{n}}$: a zonoid $\Rightarrow \boldsymbol{\mathcal{P}(K) \geq 4^n/n!}$.

[Barthe and Fradelizi, Amer. J. Math. (2013)]

Sharp lower bounds of $\boldsymbol{\mathcal{P}(K)}$ under another symmetry. (Details will be described later.)

[Artstein-Avidan, Karasev, and Ostrover, Duke Math. J. (2014)]

Viterbo's conjecture (in the context of symplectic geometry) implies Mahler's conjecture.

There are many other related resluts, however, the conjecture is still open.

Theorem [Iriyeh and S. (preprint; arXiv:1706.01749v3)]

Let $\boldsymbol{K}$ be a centrally symmetric convex body in $\mathbb{R}^{\mathbf{3}}$. Then,

$$\mathcal{P}(\boldsymbol{K}) \geq \frac{\mathbf{4^3}}{\mathbf{3!}} = \frac{\mathbf{32}}{\mathbf{3}},$$

with equality if and only if either $\boldsymbol{K}$ or $\boldsymbol{K}^{\circ}$ is a parallelepiped.

Remark

- The keys to prove the theorem are "equipartition" and "signed volume estimate".
- A simplified proof for "equipartition" is given in [Fradelizi, Hubard, Meyer, Roldán-Pensado, and Zvavitch, arXiv:1904.10765].

Motivation

- At present, we cannot solve high dimensional cases $(\boldsymbol{n} \geq 4)$, however, "signed volume estimate" is applicable. (Details will be described later.)
- To understand deeply about Mahler's conjecture, we consider a generalized problem.

Let G be a discrete subgroup of orthogonal group $O(n) \subset M_n(\mathbb{R})$.
$\mathcal{K}^n(G) = \{K \in \mathcal{K}^n; gK = K \text{ for any } g \in G\}$: the set of all G-invariant convex bodies.

Problem

Find a minimizer of minimizing problem

$$\min_{K \in \mathcal{K}^n(G)} \mathcal{P}(K).$$

Remark

- Case $G = \{E, -E\}$: Mahler's conjecture. (G-invariant iff centrally symmetric)
- Case $G = \left\{ \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \right\}$: Saint-Raymond's result.

Today, we focus the case $n = 3$.

Fact (see, e.g., [Conway–Smith, "On quaternions and Octonoins"])

Up to conjugacy, discrete subgroups of $O(3)$ are classified as 7 infinite families and 7 polyhedral groups. In Schönflies notation,

$$C_\ell, S_{2\ell}, C_{\ell h}, C_{\ell v}, D_\ell, D_{\ell d}, D_{\ell h}\ (\ell \in \mathbb{N}), \qquad T, T_d, T_h, O, O_h, I, I_h.$$

$$R_\ell := \begin{pmatrix} \cos\xi & -\sin\xi & 0 \\ \sin\xi & \cos\xi & 0 \\ 0 & 0 & 1 \end{pmatrix}, V := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \left(\ell \in \mathbb{N}, \xi := \frac{2\pi}{\ell}\right).$$

$$C_\ell := \langle R_\ell \rangle, \quad C_{\ell h} := \langle R_\ell, H \rangle, \quad C_{\ell v} := \langle R_\ell, V \rangle, \quad S_{2\ell} := \langle R_{2\ell} H \rangle,$$
$$D_\ell := \langle R_\ell, VH \rangle, \quad D_{\ell d} := \langle R_{2\ell} H, V \rangle, \quad D_{\ell h} := \langle R_\ell, V, H \rangle.$$

$$T := \{g \in SO(3); g\triangle = \triangle\}, \quad T_d := \{g \in O(3); g\triangle = \triangle\}, \quad T_h := \{\pm g; g \in T\},$$
$$O := \{g \in SO(3); gP_8 = P_8\}, \quad O_h := \{g \in O(3); gP_8 = P_8\} = \{\pm g; g \in O\},$$
$$I := \{g \in SO(3); gP_{20} = P_{20}\}, \quad I_h := \{g \in O(3); gP_{20} = P_{20}\} = \{\pm g; g \in I\},$$

$\triangle$: a regular tetrahedron (simplex), P_8: a regular octahedron, P_{20}: a regular icosahedron.

Remark

K is $C_{\ell h}$-invariant if and only if K is R_ℓ-symmetry and H-symmetry.

For a set $A \subset \mathbb{R}^n$, we denote the group of linear isometries of A by $\mathcal{O}(A) := \{g \in O(n); gA = A\}$.

Theorem. [Barthe and Fradelizi, Amer. J. Math. (2013)]

(i) Let P be a regular polytope in $\mathbb{R}^n$. Then $\mathcal{P}(K) \geq \mathcal{P}(P)$ holds for any $\mathcal{O}(P)$-invariant convex body $K \subset \mathbb{R}^n$ $(n \geq 2)$.

(ii) Let P_i be a regular polytopes or Euclidean balls in $\mathbb{R}^{n_i}$ with $n_1 + \cdots + n_k = n$. Then $\mathcal{P}(K) \geq \mathcal{P}(P_1 \times \cdots \times P_k)$ holds for any $\mathcal{O}(P_1) \times \cdots \times \mathcal{O}(P_k)$-invariant convex body $K \subset \mathbb{R}^n$ $(n \geq 2)$.

Remark

In the paper, they obtained result for equality condision of (i), and they studied also many hyperplane symmetric case.

Let $\ell \geq 3$ and $P = [-1, 1] \times$ regular ℓ-gon Q. Hence, P is a right prism with regular ℓ-gonal base. Then $\mathcal{O}([-1, 1]) \times \mathcal{O}(Q)$ is $D_{\ell h} = \langle R_\ell, V, H \rangle$. By the theorem, $\mathcal{P}(K) \geq \mathcal{P}(P)$ for any $D_{\ell h}$-invariant convex body $K \in \mathbb{R}^3$.

Theorem [Iriyeh-S. in preparation]

$\mathcal{P}(K) \geq \mathcal{P}(P)$ for any $C_{\ell h}$-invariant convex body $K \in \mathbb{R}^3$.

Known results

G	minimizer P	
D_{2h}	cube, P_8	[Saint-Raymond (1980)]
S_2	cube, P_8	[Iriyeh-Shibata] (Mahler's conjecture)
T_d	simplex	[Barthe-Fradelizi (2013)]
O_h	cube, P_8	[Barthe-Fradelizi (2013)]
I_h	P_{12}, P_{20}	[Barthe-Fradelizi (2013)]
$D_{\ell h}$ $(\ell \geq 3)$	regular ℓ-prism, regular ℓ-bipyramid	[Barthe-Fradelizi (2013)]

Main Theorem [Iriyeh-S, in preparation]

$\mathcal{P}(K) \geq \mathcal{P}(P)$ holds for G-invariant convex body K.

G	minimizer P
C_{2h}, T_h, S_6, D_{3d}	cube, P_8
$C_{\ell h}$, D_ℓ $(\ell \geq 3)$	regular ℓ-prism, regular ℓ-bipyramid

G	minimizer P
T	simplex
O	cube, P_8
I	P_{12}, P_{20}

Remaining cases and conjecture

G	minimizer P
$\{\mathrm{id.}\}$, C_{1v}, C_2, C_{2v}	simplex
C_ℓ, $C_{\ell v}$ $(\ell \geq 3)$	regular ℓ-pyramid
$S_{2\ell}$, $D_{\ell d}$ $(\ell = 2, \ell \geq 4)$	regular ℓ-antiprism, its polar

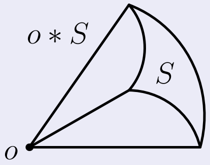
Key lemma: Signed volume estimate

In [Iriyeh-S.], we introduced "signed volume estimate".

Key lemma (signed volume estimate)

Assume that

- $K \subset \mathbb{R}^3$ is a convex body, K° is the polar of K.
- $S \subset \partial K$ with piecewise C^1 boundary $C = \partial S$.
- $S^\circ \subset \partial K^\circ$ with piecewise C^1 boundary $C^\circ = \partial S^\circ$.



Then

$$|o * S|_3 \, |o * S^\circ|_3 \geq \frac{1}{3^2} \overline{C} \cdot \overline{C^\circ}.$$

Here $o * S := \{\lambda x; x \in S, 0 \leq \lambda \leq 1\}$ is the truncated cone over S, and $\overline{C}$ is a vector valued line integral $\overline{C} := \dfrac{1}{2} \displaystyle\int_C r \times dr$, where r is a parametrization of C. $o * S^\circ$ and $\overline{C^\circ}$ are determined similarly.

Remark

If C is a curve (not necessary closed) on a plane H with $o \in H$, then $\overline{C}$ is a normal vector of H and $|\overline{C}| = |o * C|_2$.

Using smooth approximation, we can assume K is a smooth strongly convex body. Put $\Lambda(x) := \nabla \mu_K(x)$, where μ_K is the Minkowski gauge. Then $\Lambda : \partial K \to \partial K^\circ$: smooth diffeomorphism. Moreover, $x \cdot \Lambda(x) = 1$ for any $x \in \partial K$.

$$|o * S|_3 = \int_{o*S} dx = \frac{1}{3}\int_{\partial(o*S)} x \cdot n(x)\, dS(x) \qquad \text{(the divergence theorem, } \operatorname{div} x = 3\text{)}$$

$$= \frac{1}{3}\int_S x \cdot n(x)\, dS(x) \qquad (x \cdot n(x) = 0 \text{ on } \partial(o * S) \setminus S)$$

$$= \frac{1}{3}\int_S \frac{1}{|\Lambda(x)|}\, dS(x) \qquad (n(x) = \Lambda(x)/|\Lambda(x)|)$$

where $n(x)$ is the unit normal vector at x. Thus, we have

$$3^2 |o * S|_3\, |o * S^\circ|_3 = \int_S \frac{1}{|\Lambda(x)|}\, dS(x) \int_{S^\circ} \frac{1}{|\Lambda^{-1}(x^\circ)|}\, dS(x^\circ)$$

$$\geq \int_S \frac{\Lambda(x)}{|\Lambda(x)|}\, dS(x) \cdot \int_{S^\circ} \frac{\Lambda^{-1}(x^\circ)}{|\Lambda^{-1}(x^\circ)|}\, dS(x^\circ)$$

$$(\Lambda(x) \in K^\circ, \Lambda^{-1}(x^\circ) \in K, \Lambda(x) \cdot \Lambda^{-1}(x^\circ) \leq 1)$$

$$= \int_S n(x)\, dS(x) \cdot \int_{S^\circ} n(x^\circ)\, dS(x^\circ) = \frac{1}{4}\int_{\partial S} r \times dr \cdot \int_{\partial S^\circ} r^\circ \times dr^\circ.$$

(the Stokes theorem)

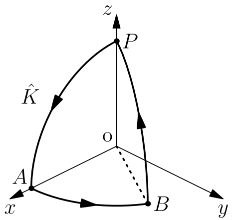
□

Using smooth approximation, we can assume K is a G-invariant smooth strongly convex body.
We use same Λ in the proof of Key Lemma.
(Put $\Lambda(x) := \nabla \mu_K(x)$, where μ_K is the Minkowski gauge. Then $\Lambda : \partial K \to \partial K^\circ$: smooth diffeomorphism. Moreover, $x \cdot \Lambda(x) = 1$ for any $x \in \partial K$.)

We denote by $\mathrm{cone}(A_1, \dots, A_k)$ the polyhedral cone generated by $A_1, \dots, A_k$.
($\mathrm{cone}(A_1, \dots, A_k) := \{\lambda_1 A_1 + \cdots + \lambda_k A_k ; \lambda_1, \dots, \lambda_k \geq 0\}$.)

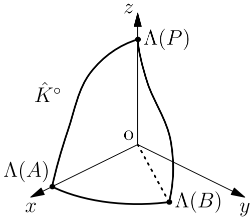
Up to linear transformation, we can assume

$$P := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, A := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, B := \begin{pmatrix} \cos 2\pi/\ell \\ \sin 2\pi/\ell \\ 0 \end{pmatrix} \in \partial K$$



$\hat{\mathcal{K}} := K \cap \mathrm{cone}(P, A, B), \quad \mathcal{S} := \partial K \cap \mathrm{cone}(P, A, B),$

$\mathcal{S}^\circ := \Lambda(S) \subset \partial K^\circ, \quad \hat{\mathcal{K}}^\circ := o * S^\circ.$



$\mathcal{C}(P, A) := \mathrm{conv}(P, A) \cap \partial K$: a curve on ∂K, from P to A.
$\mathcal{C}(A, B) := \mathrm{conv}(A, B) \cap \partial K$: a curve on ∂K, from A to B.
$\mathcal{C}(B, P) := \mathrm{conv}(B, P) \cap \partial K$: a curve on ∂K, from B to P.

Since K and K° are $C_{\ell h}$-invaritant, using Key Lemma, we have

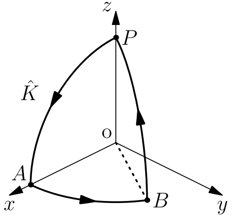
$$|K|\,|K^\circ| = 4\ell^2 |\hat{K}|\,|\hat{K}^\circ|$$
$$\geq \frac{4\ell^2}{9}\left(\overline{\mathcal{C}(P,A)} + \overline{\mathcal{C}(A,B)} + \overline{\mathcal{C}(B,P)}\right)\cdot\left(\overline{\Lambda(\mathcal{C}(P,A))} + \overline{\Lambda(\mathcal{C}(A,B))} + \overline{\Lambda(\mathcal{C}(B,P))}\right).$$

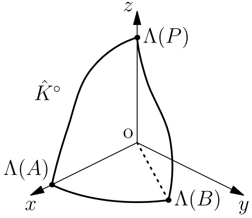
We note that, in general, $\Lambda(\mathcal{C}(P,A))$ may not be on the zx-plane, however, our signed volume estimate can apply.
Since K and K° are R_ℓ, H-symmetry, we get

$$\overline{\mathcal{C}(B,P)} = -R_\ell\overline{\mathcal{C}(P,A)},\ \overline{\Lambda(\mathcal{C}(B,P))} = -R_\ell\overline{\Lambda(\mathcal{C}(P,A))},\ \overline{\Lambda(\mathcal{C}(A,B))} \parallel \overline{\mathcal{C}(A,B)}$$

Thus, by direct calculation, we see

$$|K|\,|K^\circ| \geq \frac{4\ell^2}{9}\left(2\left(1-\cos\frac{2\pi}{\ell}\right)\overline{\mathcal{C}(P,A)}\cdot\overline{\Lambda(\mathcal{C}(P,A))} + \overline{\mathcal{C}(A,B)}\cdot\overline{\Lambda(\mathcal{C}(A,B))}\right)$$





Let H be zx-plane an $\mathbf{proj}_H$ be the projection to H. Then, we can obtain that,

$$\overline{\mathcal{C}(P,A)} \cdot \overline{\Lambda(\mathcal{C}(P,A))} = |o * \mathcal{C}(P,A)|_2\, |o * \mathbf{proj}_H(\Lambda(\mathcal{C}(P,A)))|_2.$$

Let L be $K \cap H$ and L° be the (two-dimensional) polar of L. Then, by $C_{\ell h}$ symmetry and the definition of Λ, we can check that

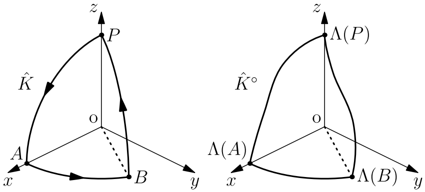
$$o * \mathcal{C}(P,A) = L \cap \mathbf{cone}(A,B), \quad o * \mathbf{proj}(o * \Lambda(\mathcal{C}(P,A))) = L^\circ \cap \mathbf{cone}(A,B).$$

Lemma (see, e.g. [Böröczky, Makai Jr, Meyer, and Reisner (2013)], [Barthe and Fradelizi (2013)], or using signed volume estimate again)

$$|L \cap \mathbf{cone}(A_1,A_2)|\, |L^\circ \cap \mathbf{cone}(B_1,B_2)| \geq \frac{1}{4}(A_2 - A_1)\cdot(B_2 - B_1).$$

Thus, we get $|L \cap \mathbf{pos}(P,A)|_2\, |L^\circ \cap \mathbf{pos}(P,A)|_2 \geq (P-A)\cdot(P-A)/4 = 1/2$.

Similarly we get $\overline{\mathcal{C}(A,B)} \cdot \overline{\Lambda(\mathcal{C}(A,B))} \geq (1 - \cos 2\pi/\ell)/2$. Consequently, we obtain the desired inequality $\mathcal{P}(K) \geq (2\ell^2)(1 - \cos 2\pi/\ell)/3$, which is the volume product of a regular ℓ-prism. $\square$



Remark

- We can prove other cases $G = D_\ell, T, O, I$ similarly.
- Using our previous result, we can treat the cases C_{2h}, T_h, S_6, D_{3d}. (In this cases, cube is a minimizer.)
- In our main theorem, we can give equality conditions also.
- Our methods can be applied to high dimensional cases $n \geq 4$ for example,

Theorem [Iriyeh-S, work in progress]

Assume $n \geq 4$. Let P be a simplex or cube in $\mathbb{R}^n$. Put $G := \{g \in SO(n); gP = P\}$. Then $\mathcal{P}(K) \geq \mathcal{P}(P)$ holds for G-invariant convex body $K \subset \mathbb{R}^n$.