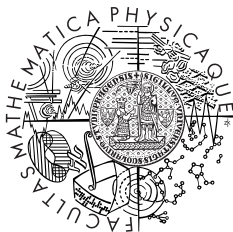


# On critical values of the distance from a subset in the plane

Jan Rataj

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joint work with Luděk Zajíček

# Distance spheres

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- ▶ even for general (compact) sets  $F \subset \mathbb{R}^d$ ,  $S_r(F)$  is a “nice” set for “many”  $r > 0$
- ▶ sufficient condition:  $r$  is a *regular value* of the distance function  $\text{dist}(\cdot, F)$

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- ▶ case  $f = d_F$  - *distance function* from a subset  $F \subset \mathbb{R}^d$ :

$$x \in \text{crit } d_F \iff 0 \in \text{conv}(\Pi_F(x) - x)$$

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Notation:  $L_F := \{r > 0 : S_r(F) \text{ is not a Lipschitz surface}\}$ ,  
 $\text{cv}(d_F) := \{r > 0 \text{ critical values of } d_F\}$ .

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$A \subset \mathbb{R}^d$ ,  $s \geq 0$

- ▶  $\overline{\mathcal{M}}^s(A) := \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\text{vol}(A + \varepsilon B)}{\varepsilon^{d-s}}$
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Remark:  $\dim_H A \leq \dim_M A$

## Definition

Let  $K \subset \mathbb{R}$  be compact and  $\alpha > 0$ . The *degree- $\alpha$  gap sum* of  $K$  is

$$G_\alpha(K) := \sum_{I \in \mathcal{G}_K} |I|^\alpha,$$

where  $\mathcal{G}_K$  is the collection of all bounded components of  $\mathbb{R} \setminus K$ .  
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## Theorem

Let  $\varepsilon > 0$  and  $K \subset [\varepsilon, \infty)$  be compact. Then TFAE:

- (i)  $K \subset L_F$  for some  $F \subset \mathbb{R}^2$  compact;
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# Idea od proof

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- ▶  $x$  critical point of  $d_F \implies (d_F)'(x, v)$  has a “jump” in direction  $v$  pointing to a nearest point of  $F$ , hence  $\text{crit } d_F$  can be covered locally by a finite number of Lipschitz graphs.

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(iii)  $\implies$  (i):

- ▶  $g(y) := \text{const } G_{\frac{1}{2}}(K \cap [\alpha, y])$ ,
- ▶  $F := \{(g(y), \pm y) : y \in K\}$ ,
- ▶  $(g(y), 0), y \in K$ , are critical points of  $d_F$ .

# A (strange) corollary

## Corollary

*Let  $D > 0$  and  $[\alpha, \beta] \subset (0, \infty)$  be given. Then there exists a finite set  $P \subset [\alpha, \beta]$  such that  $P \cap \text{cv}(d_F) \neq \emptyset$  whenever  $\emptyset \neq F \subset M$  is compact and  $\text{diam } F \leq D$ .*

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Remark:  $\text{card } P = \lfloor \text{const } D^4 (\beta - \alpha)^{-1} \beta^{-\frac{3}{2}} \rfloor + 3$

## Second result

### Theorem

Let  $A \subset (0, \infty)$ . Then TFAE:

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$$\int_0^\infty G_{\frac{1}{2}}(\bar{A} \cap [\varepsilon, \infty)) \sqrt{\varepsilon} \, d\varepsilon < \infty.$$

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If  $F \subset \mathbb{R}^2$  is compact then  $\mathcal{M}^{\frac{4}{5}}(\text{cv}(d_F)) = 0$ . Hence,  
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Remark: There exists a compact set  $F \subset \mathbb{R}^2$  with  $G_{\frac{4}{5}}(\text{cv}(d_F)) = \infty$  (hence,  $\dim_M(\text{cv}(d_F)) = \frac{4}{5}$ ).

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$M$  complete, connected Riemannian manifold,  $F \subset M$  closed,  
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$p \in M$  is a critical point of  $d_F$  if for any chart  $(U, \phi)$  covering  $p$ ,  $\phi(p)$  is a critical point of  $d_F \circ \phi^{-1}$ . Equivalently,  $p$  is critical for  $d_F$  if for any unit vector  $v \in T_p M$  there exists a shortest segment  $\gamma : [0, r] \rightarrow M$  with  $\gamma(0) = p$ ,  $\gamma(r) \in F$  such that  $\angle(v, \gamma'(0)) \leq \frac{\pi}{2}$ .  
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## Theorem

Let  $M$  be a connected complete two-dimensional Riemannian manifold,  $\emptyset \neq F \subset M$  compact and  $\varepsilon > 0$ . Then  $\text{cv}(d_F) \cap [\varepsilon, \infty)$  is a  $BT_{\frac{1}{2}}$ -set. In particular,  $\mathcal{M}^{\frac{1}{2}}(\text{cv}(d_F)) = 0$ .

## Basic tool: Ferry's inequality

Recall: If  $M = \mathbb{R}^2$  then

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*For any  $\emptyset \neq K \subset M \setminus F$  compact there exists  $C > 0$  such that for any  $p, q \in K$  that are critical for  $d_F$ ,*

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The proof uses the Toponogov comparison theorem ( $M$  is locally a space of curvature bounded from below).

# Remarks and questions

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- ▶ in dimension  $d > 3$  -  $cv(d_F)$  can have positive Lebesgue measure
- ▶ properties for special sets or set classes are of importance (self-similar fractal sets, Brownian path...)