Almost Euclidean sections of convex bodies

Convex, discrete and integral Geometry, Jena

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Based on joint work with P.Valettas and K. Tihkomirov

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- John (1949) "Isotropic Characterization" when a symmetric convex body has the property that $B_2^n \subseteq K$ and if $\mathcal{E} \subseteq K$ then $|\mathcal{E}| < |B_2^n|$.

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- Dvoretzky (1961). For every $\varepsilon > 0$ and for every n and every $K \in \mathbb{R}^n$ for $k \simeq \varepsilon \frac{\sqrt{\log n}}{\log \log n}$ there exists a $a >$ and a subspace $F \in G_{n,k}$ such that

$$aB_2^F \subseteq K \cap F \subseteq (1+\varepsilon) a B_2^F.$$

History-Dvoretzky's theorem

- V. Milman (1971) (Random) For every $\varepsilon > 0$ and for every n and every $K \in \mathbb{R}^n$ for $k \simeq \frac{\varepsilon^2}{\log 1/\varepsilon} \log n$ there exists a $a >$ and a subspace $F \in G_{n,k}$ such that

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- Schechtman (2006) (Existence) $k \simeq \ \frac{\varepsilon}{\log^2(1/\varepsilon)} \log n$.
- P.-Valettas (2018) (Random) $k \simeq \ \frac{\varepsilon}{\log(1/\varepsilon)} \log n$.

V. Milman: The concentration of measure

- Let G be a standard n-dimensional Gaussian random vector. Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an L-Lipschitz function. Then for $t > 0$,

$$\mathbb{P}\left(|f(G) - \mathbb{E}f(G)| \geq t\mathbb{E}f(G)\right) \leq 2\exp\left\{-\frac{t^2}{2}\left(\frac{\mathbb{E}f}{L}\right)^2\right\}.$$

- if $f := \|\cdot\|_K$ then $L := b(K) := \max_{\theta\in S^{n-1}} \|\theta\|_K$.
- Let K be a symmetric convex body in $\mathbb{R}^n$. We write

$$\kappa(K) := \left(\frac{\mathbb{E}\|G\|_K}{b(K)}\right)^2$$

for the 'Dvoretzky number" of K.

V. Milman: The concentration of measure

- Concentration again for $f := \|\cdot\|_K$:

$$\mathbb{P}\left(|f(G) - \mathbb{E}f| \geq \varepsilon \mathbb{E}f\right) \leq 2\exp\left\{-\frac{\varepsilon^2}{2}\left(\frac{\mathbb{E}f}{L}\right)^2\right\}, \varepsilon > 0$$

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- a net argument implies that for a **random** $F \in G_{n,k}$, where $k \simeq \frac{\varepsilon^2}{\log 1/\varepsilon}\kappa(K)$

$$(1-\varepsilon)aB_2^F \subseteq K \cap F \subseteq (1+\varepsilon)aB_2^F,$$

where $a = c_n \mathbb{E}f$.

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  where $a = c_n \mathbb{E}f$.
- In John's position $\kappa(K) \geq \log n$ (Sharp for B_∞^n).

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- Consider $\|x\|_4 := \left(\sum_{i=1}^n |x_i|^4\right)^{\frac{1}{4}}$. Then

$$\mathbb{P}\left(\left|\|G\|_4 - \mathbb{E}\|G\|_4\right| \geq \varepsilon \mathbb{E}\|G\|_4\right) \leq 2\exp\{-c\sqrt{n}\sqrt{\varepsilon}\}, \varepsilon \in (0,1)$$

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- Consider $\|x\|_\infty := \max_{i \leq n} |x_i|$. Then

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Main result: Concentration - P.-Valettas

- If K is a symmetric convex body in $\mathbb{R}^n$ we write

$$A(K,\varepsilon) := \mathbb{P}\left(\left|\|G\|_K - \mathbb{E}\|G\|_K\right| \geq \varepsilon \mathbb{E}\|G\|_K\right), \varepsilon \in (0,1)$$

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- (Tikhomirov, 2017) The 1-uncoditional case in the ℓ-position.

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- When $\varepsilon \in (0,1)$, $A(K,\varepsilon)$ is NOT stable!
- For every symmetric convex body in $\mathbb{R}^n$ there exists a $K_1 \in \mathbb{R}^n$ such that

$$d_G(K,K_1) \leq 2$$

and

$$A(K_1,\varepsilon) \geq c\exp\{-c'\varepsilon^2\kappa(K)\}, \varepsilon \in (0,1).$$

Elements of the proof- Superconcentration

- *Superconcentration:* Use of "Talagrand's" $L_1 - L_2$ inequality.

$$\mathrm{var}(f) \leq C \sum_{i=1}^{n} \frac{\|\partial_i f\|_2^2}{1 + \log\left(\frac{\|\partial_i f\|_2}{\|\partial_i f\|_1}\right)}.$$

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- Balance the $\|\partial_i f\|_1$. (Borsuk-Ulam) There exists a diagonal map Λ such that

$$\|\partial_i (f \circ \Lambda)\|_1 = \|\partial_j (f \circ \Lambda)\|_1.$$

Elements of the proof- The unconditional Constant

- Then (apply for $f := \|\cdot\|_K$) one can show that

$$A(K,\varepsilon) \leq C \exp\left(-c'\varepsilon \log \frac{n}{\mathcal{U}(K)^2}\right), \varepsilon \in (0,1)$$

where $\mathcal{U}(K)$ is the "unconditional constant" of K.

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  where $\mathcal{U}(K)$ is the "unconditional constant" of K.
- In general $\mathcal{U}(K) \leq \sqrt{n}$.

Elements of the proof- The dichotomy of Alon-Milman

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Elements of the proof- The dichotomy of Alon-Milman

- Alon-Milman , Talagrand. Dichotomy: Either $k(\kappa)$ is n^{γ} for some $\gamma \in (0,1)$ or
- or there exists a subspace F of dimension n^{δ} such that $\mathcal{U}(K \cap F) < n^{\eta}$ with $\eta << \frac{\delta}{2}$.

The main result-Dvoretzky type

- **Theorem:** (P. -Valettas, 2018) For every symmetric convex body K in $\mathbb{R}^n$ there exists a linear image $\tilde{K}$ such that for every $\varepsilon > 0$ and $k \simeq \frac{\varepsilon}{\log 1/\varepsilon} \log n$ a **random** subspace of dimension k satisfies

$$a(1-\varepsilon)B_2^F \subseteq \tilde{K} \cap F \subseteq (1+\varepsilon)aB_2^F,$$

where $a := c_n \mathbb{E}\|G\|_{\tilde{K}}$.

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- **Theorem:** (Tikhomirov (2014)) Let $a := c_n \mathbb{E}\|G\|_\infty$. If for a random subspace F

$$a(1-\varepsilon)B_2^F \subseteq \tilde{B_\infty} \cap F \subseteq (1+\varepsilon)aB_2^F,$$

then $k \leq c\frac{\varepsilon}{\log 1/\varepsilon} \log n$.

The small ball problem

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- $$\mathbb{P}\left(\|G\|_\infty \leq (1-\varepsilon)\mathbb{E}\|G\|_\infty\right) \leq c_1 e^{-e^{c_2 \varepsilon \log n}}.$$
- The concentration inequality gives

$$\mathbb{P}\left(\|G\|_K \leq (1-\varepsilon)\mathbb{E}\|G\|_K\right) \leq c_1 e^{-\varepsilon^2 \kappa(K)}.$$

Comparison of the behaviour

- Consider the norm $\|G\|_4$. Then one can check that

$$\mathbb{P}\left(\|G\|_4 \geq t\mathbb{E}\|G\|_4\right) \leq e^{-ct^2\sqrt{n}}, t \geq 2$$

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Main Result, Small ball for norms

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- *P. K. Tikhomirov P. Valettas* Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$.
- If $\|\cdot\|$ is 1–unconditional, in the ℓ–position, then for any $\delta \in (0, 1/2]$ we have
$$\mathbb{P}\left\{\|G\| \leq \delta \mathbb{E}\|G\|\right\} \leq 2\exp\left(-n^{1-C\delta^2}\right).$$

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$$\mathbb{P}\left\{\|G\| \leq \delta \mathbb{E}\|G\|\right\} \leq 2\exp\left(-n^{1-C\delta^2}\right).$$
- If $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$, then there exists a linear invertible map T such that for every $\delta \in (0, 1/2]$ we have
$$\mathbb{P}\left\{\|TG\| \leq \delta \mathbb{E}\|TG\|\right\} \leq 2\exp\left(-n^{1/4-C\delta^2}\right).$$

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$$\mathbb{P}\left\{\|G\| \leq (1-\varepsilon)\mathbb{E}\|G\|\right\} \leq 2\exp\left(-n^{c\varepsilon}\right).$$
- In the general case, there exists an invertible linear map T such that for any $\varepsilon \in (0, 1/2)$ we have
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Corollaries - One-sided Dvoretzky's theorem

- Let K be a symmetric convex body in $\mathbb{R}^n$. Then there exists a linear image of K, $\tilde{K}$ such that for every $1 \leq m \leq n^{\frac{1}{5}}$ for a **random** k-dimensional subspace $F \in G_{n,k}$

$$cW(\tilde{K})B_2^F \subseteq P_F\tilde{K}.$$

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- Moreover, if K is 1-uncoditional in the minimal mean width position, for any $\varepsilon \in (0, \frac{1}{2})$ and $k \leq cn^{1-c\varepsilon^2}$ the random k-dimensional F satisfies

$$\frac{c}{\varepsilon^2}W(\tilde{K})B_2^F \subseteq P_F\tilde{K}.$$

Corollaries - Concentration of Quermassintegrals

- Let K be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Let

$$Q_k(K) := \left(\frac{1}{|B_2^k|} \int_{G_{n,k}} |P_F K| dF \right)^{\frac{1}{k}}.$$

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- Note that $Q_1(K) := W(K)$ and $Q_n(K) := \mathrm{v.r.}(K)$.
- The Alexandrov's inequality implies that

$$Q_n(K) \leq Q_{n-1}(K) \leq \cdots \leq Q_1(K).$$

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- Moreover, if K is 1-uncoditional in the minimal mean width position, for any $\varepsilon \in (0, \frac{1}{2})$ and $k \leq cn^{1-c\varepsilon^2}$,

$$Q_k(K) \geq c\varepsilon^2 Q_1(K).$$

Elements of the proof

- Small deviation inequality for convex functions (Ehrhard). If f is convex,
$$\mathbb{P}\left(f(G) \leq \mathbb{E}f(G) - \varepsilon\sqrt{\mathrm{var}(f)}\right) \leq 2e^{-c\varepsilon^2}.$$

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$$P_t(f)(x) := \mathbb{E}f\left(e^{-t}x + \sqrt{1-e^{-2t}}G\right).$$
- $P_t f$ is again convex, $\mathbb{P}_t f = \mathbb{E}f$.

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- Optimize with respect to t and ε.

Thank you!

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