An Optimal Plank Theorem

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What is a Plank?

A plank in a vector space X is the region bounded by two parallel hyperplanes.

Tarski Plank Problem

If an n-dimensional convex body is covered by a collection of planks, then the sum of the widths of the planks should be at least the minimal width of the convex body they cover.

Tarski's plank problem

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- Bang (1951): arbitrary convex bodies.

Affine Plank Problem

Bang (1951) also asked whether the widths of the planks could be measured with respect to the convex body that it is covered.

- Ball (1990) solved this affine version of the plank problem for the most interesting case: symmetric convex body.

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- Generalization of Hahn-Banach Theorem.
- A sharp quantitative version of the uniform boundedness principle.
- A geometric pigeon-hole principle.

Plank in normed spaces

A plank in a normed space X is a region of the form

$$\{x \in X : |\phi(x) - m| \leq w\}$$

where ϕ is a linear functional on X^* of norm 1, m a real number, and w is a positive number. The number w is called the half-width of the plank.

Ball's Plank theorem

Theorem (The Plank Theorem)

For any sequence $(\phi_k)_{k=1}^{\infty}$ *of norm one functionals on a real Banach space* X, $(m_k)_{k=1}^{\infty}$ *a sequence of real numbers and non-negative numbers* $(t_k)_{k=1}^{\infty}$ *satisfying*

$$\sum_{k=1}^{\infty} t_k < 1,$$

there exists a unit vector x *in* X *for which*

$$|\phi_j(x) - m_j| > t_j$$

for every j.

Ball's Plank theorem

The Plank Theorem is obviously sharp in the sense that the unit ball of X can be covered by n non-overlapping parallel planks whose half-widths add up to 1.

Our Problem

We are interested in this problem in the case that the planks covering the convex body are symmetric about the origin:

$$\{x \in X : |\phi(x)| \leq w\}$$

where ϕ is a linear functional on X^* of norm 1 and w is a positive number.

Our Problem

Theorem (The Plank Theorem)

For any sequence $(\phi_k)_{k=1}^{\infty}$ of norm one functionals on a (real) Banach space X and non-negative numbers $(t_k)_{k=1}^{\infty}$ satisfying

$$\sum_{k=1}^{\infty} t_k < 1,$$

there exists a unit vector x in X for which

$$|\phi_j(x)| > t_j$$

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- For other spaces we expect to be able to improve upon this condition. Hilbert Spaces?
- Ball proved that for *complex* Hilbert spaces it is possible to *beat* any sequence for which $\sum_k t_k^2 = 1$.

Complex Plank Theorem (2001)

Theorem (Complex Plank Theorem)

For any sequence $v_1, v_2, \ldots, v_n$ *of unit vectors in a complex Hilbert space* H *and positive real numbers* $t_1, t_2, \ldots, t_n$ *satisfying*

$$\sum_{k=1}^{n} t_k^2 = 1$$

there exists a unit vector $z \in H$ *such that*

$$|\langle v_k, z\rangle| \geq t_k$$

for all k*.*

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Complex Plank Theorem

Theorem (Complex Plank Theorem for same width)

For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a complex Hilbert space H there exists a unit vector $z \in H$ such that

$$|\langle v_k, z\rangle| \geq \frac{1}{\sqrt{n}}$$

Real Hilbert spaces

What happens for *real* Hilbert spaces?

Real Hilbert spaces

This is not possible. Consider $2n$ vectors $v_1, v_2, \ldots, v_{2n}$ in $\mathbb{R}^2$ equally spaced around the circle: (n vectors and their negatives). For any unit vector v in $\mathbb{R}^2$ there is a i such that

$$|\langle v_i, v\rangle| \leq \sin(\pi/2n).$$

Fejes Tóth's zone conjecture

This simple statement is connected to a conjecture by Fejes Tóth that was positively answered, about two years ago, by Jiang and Polyanskii.
A zone of with w is the set of points in the unit sphere at spherical distance $w/2$ of a given great circle. In 1973, Fejes Tóth conjectured that if a collection of zones of equal width covers the unit sphere then the width of the zones should be at least π/n.

Zone

A zone of spherical width w associated to the great circle $\mathbb{S}^2 \perp v$, for a given unit vector v, is the set given by

$$\{x \in \mathbb{S}^2 : |\langle v, x \rangle| \leq \sin(w/2)\}$$

With this notation, Fejes Tóth conjecture can be restated and generalized as an optimal plank problem for real Hilbert spaces.

Main Theorem

Theorem (Z. Jiang and A. Polyanskii 2017 ;O. Ortega 2019)

For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a real Hilbert space H, there exists a unit vector $v \in H$ such that

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for all $i \in \{1, 2, \ldots, n\}$.

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Z. Jiang and A. Polyanskii used a completely different approach.

Strategy

The basic strategy in the proof of the main theorem is the strategy followed by Ball in the proof the Complex Plank Theorem, but there is a fundamental difference.

Strategy

Fundamental difference: the main ingredient of the proof of the Complex Plank Theorem has no analogue in the real case.

Strategy

Ball studies the behaviour of a complex polynomial locally around 1 and, with the aid of the maximum modulus principle, manages to jump away from 1 to a point in the unit disk where this polynomial has large absolute value.

Strategy

In contrast, the proof of the main theorem here relies on extremal properties of trigonometric polynomials to produce this jump.

Rescaled version

Theorem

For any sequence $v_1, v_2, \ldots, v_n$ of unit vectors in a real Hilbert space H, there exists a vector $v \in H$ of norm $\sqrt{n}$ for which

$$|\langle v_k, v \rangle| \geq \sqrt{n} \sin(\pi/2n)$$

for all k.

Inverse Eigenvectors: Motivation

The treatment and adaptations of Ball's approach to the Complex Plank Theorem and the definition of inverse eigenvectors that we present here are due to Ambrus. He used this method to solve the strong polarization problem in the planar case (Ambrus UCL PhD thesis 2009).

Inverse Eigenvectors: Motivation

- Our problem: finding a vector v of norm $\sqrt{n}$ which has large inner product with all the vectors $v_1, v_2, \ldots, v_n$.

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- Our problem: finding a vector v of norm $\sqrt{n}$ which has large inner product with all the vectors $v_1, v_2, \ldots, v_n$.
- Obvious candidate: a vector v of norm $\sqrt{n}$ for which $\min_k |\langle v_k, v \rangle|$ is maximal. Difficult to manipulate...

Instead...

- We choose a unit vector v of norm $\sqrt{n}$ for which the product $\prod_i |\langle v_i, v\rangle|$ is maximal.

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- Hope that the factors are large enough to get the desired inequality.
- Advantage: we can use simple analytic tools to study the points for which $\prod_i |\langle v_i, v\rangle|$ is locally extremal.

Structure of extremal points

Proposition (G. Ambrus 2009)

Let $v_1, v_2, \ldots, v_n$ be a sequence for unit vectors in a real Hilbert space H. Suppose that v is vector of norm $\sqrt{n}$ chosen so as to maximize

$$\prod_{k=1}^{n} | \langle v_k, v \rangle |.$$

Then,

$$v = \sum_{k=1}^{n} \frac{1}{\langle v_k, v \rangle} v_k$$

Structure of extremal points

Denote by H the Gram matrix associated to the sequence of unit vectors $(v_k)_{k=1}^n$, that is, $H_{ij} = \langle v_i, v_j \rangle$, and let w be the vector in $\mathbb{R}^n$ given by

$$w_k = \frac{1}{\langle v_k, v \rangle}$$

for all k. Then w satisfies

$$(Hw)_j = \sum_{i=1}^n h_{ji} w_i = \langle v_j, \sum_{i=1}^n w_i v_i \rangle = \langle v_j, v \rangle = \frac{1}{w_j}.$$

Therefore, w satisfies the following equation $Hw = w^{-1}$ where w^{-1} is define as the inverse of the vector w componentwise, i.e.

$$w^{-1} = \left(\frac{1}{w_1}, \dots, \frac{1}{w_n} \right).$$

Inverse Eigenvectors

Definition (G. Ambrus 2009)

Let M be a $n \times n$ matrix. We say that w is an *inverse eigenvector* of M if

$$Mw = w^{-1}$$

Connection with Inverse Eigenvectors

Given an inverse eigenvector w of H, one can set

$$v = \sum_{k=1}^{n} w_k v_k.$$

Then, v would satisfy

$$v = \sum_{k=1}^{n} \frac{1}{\langle v_k, v \rangle} v_k$$

and

$$w_k = \frac{1}{\langle v_k, v \rangle}$$

Theorem in terms of Inverse Eigenvectors

Theorem (O. Ortega 2019)

Let H be a real Gram matrix. Then, there exists an inverse eigenvector w of H for which

$$\|w\|_\infty \leq n^{-\frac{1}{2}} \csc(\pi/2n).$$

Finding an Inverse Eigenvector

To find a suitable inverse eigenvector w notice that w defined as before is a local extremal point for the function

$$\prod_{k=1}^{n} |\langle v_k, v\rangle|,$$

which in terms of w is given by

$$\prod_{k=1}^{n} \frac{1}{|w_k|}$$

subject to the constraint

$$\|v\|^2 = \left\|\sum w_k v_k\right\|^2 = w^\top H w = n$$

Finding an Inverse Eigenvector

- Find a vector w to minimize $\prod |w_k|$ subject to the constrain $w^\top H w = n$.
- This minimum is always 0!
- Choose u so as to maximize $\prod |u_k|$ subject to the constrain $u^\top H^{-1} u = n$
- Hope that the maximum would be converted into a local minimum of the original problem via the natural bijection between the inverse eigenvectors of H and H^{-1}: If u is an inverse eigenvector of H^{-1}, then $w = u^{-1}$ is an inverse eigenvector of H.

Finding an Inverse Eigenvector

Lemma

Suppose that H is a real Gram matrix and u is a vector for which

$$\prod_{k=1}^{n} |u_k|$$

is locally extremal subject to the condition

$$u^\top H^{-1} u = n.$$

Then, u is an inverse eigenvector for H^{-1}.

Finding an Inverse Eigenvector

- The lemma yields a vector u for which $\prod |u_k|$ is *maximal* subject to the constrain $u^\top H^{-1} u = n$. Set $w = u^{-1}$. Thus, w is an inverse eigenvector of H.

Finding an Inverse Eigenvector

- The lemma yields a vector u for which $\prod |u_k|$ is *maximal* subject to the constrain $u^\top H^{-1} u = n$. Set $w = u^{-1}$. Thus, w is an inverse eigenvector of H.
- Maximal condition of u: if c is a vector such that $\prod |c_k| = 1$, then

$$\prod |c_k u_k| = \prod |u_k|$$

and therefore

$$\sum_{jk} c_j u_j H^{-1}_{jk} c_k u_k \geq n$$

Proving the bound

The problem is to show $\|w\|_\infty \leq n^{-\frac{1}{2}} \csc(\pi/2n)$.

Final Transformation.

Define a matrix M by

$$M_{ij} = w_j H_{jk} w_k$$

for all j, k. M is a positive matrix and its inverse is given by

$$M_{ik}^{-1} = u_j H_{jk}^{-1} u_k.$$

Observe that

$$m_{kk} = |w_k|^2.$$

If we denote by $\mathbf{1}$ the vector whose entries are all equal to 1, then

$$\begin{aligned} (M\mathbf{1})_j &= \sum_k w_j H_{jk} w_k \\ &= w_j \sum_k H_{jk} w_k \\ &= w_j (Hw)_j = 1 \end{aligned}$$

Final Transformation.

Lemma (O. Ortega 2019)

Suppose that M is a symmetric positive matrix satisfying

- $M\mathbf{1} = \mathbf{1}$, *and*
- *whenever c is a vector such that $\prod |c_k| = 1$, then*

$$c^\top M^{-1} c \geq n.$$

Then $m_{kk} \leq n^{-1} \csc^2(\pi/2n)$ for all k.

Proof

Notice that if we let $c = Mb$ then the second condition of the lemma states that if $\prod |(Mb)_j| = 1$,

$$b^\top Mb \geq n.$$

Let us assume, for a contradiction, that one of the diagonal entries is too large. Thus, without loss of generality, assume that for the first diagonal entry

$$m_{11} > \frac{1}{n \sin^2(\pi/2n)}$$

Let $\mathcal{E}$ be the ellipsoid defined by the equation $b^\top M b = n$, i.e.

$$\mathcal{E} = \{b : b^\top M b = n\}$$

In order to get a contradiction, we need to find a vector v *inside* the ellipsoid $\mathcal{E}$ for which the product $\prod (Mv)_k$ is greater of equal to 1.

The proof consists of looking at a 2-dimensional slice of the ellipsoid $\mathcal{E}$ passing through the point $\mathbf{1}$. Given a vector $v \in \mathcal{E}$ orthogonal to $\mathbf{1}$, denote by H_v the subspace spanned by v and $\mathbf{1}$.

Denote by $\mathcal{E}_v$ the 2-dimensional ellipse we get by intersecting $\mathcal{E}$ and H_v,

$$\mathcal{E}_v = \mathcal{E} \cap H_v$$

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The natural candidate as to get a better estimate for m_{11} would be

$$\begin{aligned} H &= \{(x, y, \ldots, y) \in \mathbb{R}^n, y \in \mathbb{R}\} \\ &= \operatorname{span}\{e_1, \mathbf{1}\} \\ &= \operatorname{span}\{v_1, \mathbf{1}\} = H_{v_1} \end{aligned}$$

where

$$v_1 = \frac{ne_1 - \mathbf{1}}{\sqrt{nm_{11} - 1}}$$

Consider the following vector

$$v^{(\alpha)} = -\sqrt{\alpha} v_1$$

where

$$\alpha = \frac{\cot^2(\pi/2n)}{nm_{11} - 1}$$

Notice is that $\alpha \in (0, 1)$. For each $\theta \in [0, 2\pi)$, define

$$v_\theta^{(\alpha)} = \cos\theta \ \mathbf{1} + \sin\theta \ v^{(\alpha)}.$$

Notice that $v_\theta^{(\alpha)}$ is just a parametrisation of a 2- dimensional ellipse strictly *inside* $\mathcal{E}$ that lies on the subspace H.

In fact, if $\theta \neq 0$ or π,

$$\begin{aligned} {v_\theta^{(\alpha)}}^\top M v_\theta^{(\alpha)} &= n \cos^2 \theta + \alpha n \sin^2 \theta \\ &= n(\cos^2 \theta + \alpha \sin^2 \theta) < n \end{aligned}$$

and ${v_\theta^{(\alpha)}}^\top M v_\theta^{(\alpha)} = n$ if and only if $\theta = 0$ or π.

We want to show that there is a $\theta \in [0, 2\pi) \setminus \{0, \pi\}$ such that

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Substituting

$$\prod_{j=1}^{n} (Mv_\theta^{(\alpha)})_j = \prod_{j=1}^{n} \Big(\cos\theta + (Mv_k^{(\alpha)})_j \sin\theta \Big)$$

The product happens to be a trigonometric polynomial of degree at most n!

Define the trigonometric polynomial $T_{v^{(\alpha)}}$ by

$$\begin{aligned} T_{v^{(\alpha)}}(\theta) &= \prod_{j=1}^{n}(Mv_\theta^{(\alpha)})_j \\ &= \prod_{j=1}^{n}\Big(\cos\theta + (Mv^{(\alpha)})_j \sin\theta\Big)_j \end{aligned}$$

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Thus, we want to show that there exists $\theta \neq 0$ and π such that $|T_{v^{(\alpha)}}(\theta)| \geq 1$.

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Hence, $T_{v(\alpha)}(\theta)$ and $\cos(n\theta)$ most meet in many points!

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How many?

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How many?

$$Q(\theta) = T_{v(\alpha)}(\theta) - \cos n\theta$$

Counting roots of Q

θ	$\cos n\theta$	$Q(\theta)$
0	1	0
π/n	-1	-
$2\pi/n$	1	$+$
$\vdots$	$\vdots$	$\vdots$
$(n-1)\pi/n$	1	$+$
π	-1	0
$(n+1)\pi/n$	1	$+$
$\vdots$	$\vdots$	$\vdots$
2π	1	$+$

By the Intermediate Value Theorem we know we have a root of Q in the intevals:

$(0, \pi/n)$	no
$(\pi/n, 2\pi/n)$	yes
$(2\pi/n, 3\pi/n)$	yes
$\vdots$	$\vdots$
$((n-2)\pi/n, (n-1)\pi/n)$	yes
$((n-1)\pi/n, \pi)$	no
$(\pi, (n+1)\pi/n)$	no
$((n+1)\pi/n, (n+2)\pi/n)$	yes
$\vdots$	$\vdots$
$((2n-2)\pi/n, (2n-1)\pi/n)$	yes
$((2n-1)\pi/n, \pi$	no

At least $2n-4$, plus 0 and π, $2n-2$.

So the polynomial Q has at least $2n - 2$ roots. Is this enough?

So the polynomial Q has at least $2n-2$ roots. Is this enough?

No

However...

Expanding the product we get

$$T_{v^{(\alpha)}}(\theta) = \cos^n \theta + \sum_j (Mv^{(\alpha)})_j \cos^{n-1} \theta \sin \theta + \sin^2 \theta \ \psi(\theta)$$

where ψ is a trigonometric polynomial of degree at most $n-2$. On the other hand,

$$\sum_j (Mv^{(\alpha)})_j = \mathbf{1}^\top Mv^{(\alpha)} = \mathbf{1}^\top v^{(\alpha)} = 0$$

and therefore,

$$T_{v^{(\alpha)}}(\theta) = \cos^n \theta + \sin^2 \theta \ \psi(\theta). \tag{1}$$

However...

It is easy to see that $\cos n\theta$ is of the form (1); thus, taking the difference of $T_{v(\alpha)}(\theta)$ and $\cos n\theta$ we get

$$\begin{aligned} Q(\theta) &= T_{v(\alpha)}(\theta) - \cos n\theta \\ &= \sin^2\theta\ \psi(\theta) \end{aligned}$$

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Are we done?

No yet, We still need to find at least one root more.

Recall that

$$T_{v^{(\alpha)}}(\theta) = \prod_{j=1}^{n} \left(\cos\theta + (Mv^{(\alpha)})_j \sin\theta \right)$$

Notice that

$$(Mv^{(\alpha)})_1 = -\sqrt{\alpha}(Mv_1)_1 = \cot\left(\frac{\pi}{2n}\right)$$

and so the first factor of $T_{v^{(\alpha)}}$ is equal to 0 if and only if

$$\cos\theta = \cot\left(\frac{\pi}{2n}\right)\sin\theta$$

which happens if and only if $\theta = \frac{\pi}{2n}$ or $\pi + \frac{\pi}{2n}$. Hence, $T_{v^{(\alpha)}}$ has roots $\theta = \frac{\pi}{2n}$ and $\pi + \frac{\pi}{2n}$. So we have a couple of roots more! a contradiction!

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Q. E. D.

Thank you!