Optimal comparison of weak and strong moments of random vectors with applications

(joint work with Rafał Latała)

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Theorem (Latała, N., 2019)

Let X be a random vector in $\mathbb{R}^n$ and $\emptyset \neq T \subseteq \mathbb{R}^n$. Then for $p \geq 2$

$$\left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p\right)^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}}\sup_{t\in T}\left(\mathbb{E}|\langle t, X\rangle|^p\right)^{1/p}.$$

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Remark. This result is optimal (up to a universal constant) and equality is achieved for any rotationally invariant random vector X and $T = B_2^n$.

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Remark. This result is optimal (up to a universal constant) and equality is achieved for any rotationally invariant random vector X and $T = B_2^n$.

Proof inspired by the Welch bound proof of Datta, Stephen and Douglas (2012).

Application I - p-summing constant.

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Application I - p-summing constant.

Theorem (Latała, N., 2019) (strong vs. weak moments)

Let X be a random vector in $(\mathbb{R}^n, \|\cdot\|)$. Then for $p \geq 2$

$$(\mathbb{E}\|X\|^p)^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}} \sup_{\|t\|_* \leq 1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}.$$

Application I - p-summing constant.

Corollary (Latała, N., 2019)

Let $(F, \|\cdot\|)$ be a Banach space of dimension n. Then for any vectors $x_1, \ldots, x_l \in F$ we have

$$\left(\sum_{j=1}^{l} \|x_j\|^p\right)^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}} \sup_{\|t\|_* \leq 1} \left(\sum_{j=1}^{l} |\langle t, x_j\rangle|^p\right)^{1/p}.$$

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The best constant $\pi_p(F)$ in this inequality is called the p-summing constant of F. We have $\pi_p(l_2^n) = (\mathbb{E}|U_1|^p)^{-1/p} \approx \sqrt{\frac{n+p}{p}}$.
Therefore

$$\pi_p(F) \leq c\pi_p(l_2^{\dim F}).$$

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Question. Is it true that $\pi_p(F) \leq \pi_p(l_2^{\dim F})$?

Application II - concentration of measure theory.

Theorem (Latała, N., 2019)

Every centered log-concave probability measure on $\mathbb{R}^n$ satisfies the optimal concentration inequality in the sense of Latała and Wojtaszczyk with a constant $\sim n^{5/12}$.

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Remark. Previous best bound $\sim n^{1/2}$ was due to Latała.

Basic linear algebra facts.

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Lemma 1 (rank factorization)

Suppose A is a $k \times l$ matrix of rank at most n. Then A can be written as a product $A = TX$, where T is $k \times n$ and X is $n \times l$:

$$A = TX = \underbrace{\begin{bmatrix} — & t_1 & — \\ & \vdots & \\ — & t_k & — \end{bmatrix}}_{n} \cdot \left.\begin{bmatrix} | & & | \\ x_1 & \cdots & x_l \\ | & & | \end{bmatrix}\right\} n = (\langle t_i, x_j \rangle)_{i \le k, j \le l}$$

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Proof. There exist vectors $v^{(1)}, \ldots, v^{(n)}$ such that every column a of A can be written as

$$a = \sum_{s=1}^{n} v^{(s)} \lambda_s.$$

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Lemma 2 (rank of Hadamard product), Peng-Waldron, 2002

Suppose $A = (a_{ij})$ is a $k \times l$ matrix of rank at most n. Let m be a positive integer. Then $A^{\circ m} := (a_{ij}^m)$ has rank at most $\binom{n+m-1}{m}$.

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$$a_i^m = \sum_{s_1, s_2, \dots, s_m = 1}^{n} v_i^{(s_1)} v_i^{(s_2)} \cdot \ldots \cdot v_i^{(s_m)} \lambda_{s_1} \lambda_{s_2} \cdot \ldots \cdot \lambda_{s_m}.$$

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We conclude that

$$a^m \in \operatorname{span} \left\{ (v_i^{(s_1)} \cdot \ldots \cdot v_i^{(s_m)})_{i=1,\ldots,k}, \;\; 1 \le s_1 \le \ldots \le s_m \le n \right\}.$$

Lemma 3 (case $p = 2$)

Let X be a random vector in $\mathbb{R}^n$ and let us take $\emptyset \neq T \subseteq \mathbb{R}^n$. Then

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Let C be the covariance matrix of a symmetric, bounded and non-degenerate X. We can assume $\sup_{t \in T} \mathbb{E}|\langle t, X\rangle|^2 = 1$ and

$$\begin{aligned} T = \{t \in \mathbb{R}^n : \; \mathbb{E}|\langle t, X\rangle|^2 \leq 1\} &= \{t \in \mathbb{R}^n : \; \langle Ct, t\rangle \leq 1\} \\ &= \{t \in \mathbb{R}^n : \; |C^{1/2} t| \leq 1\}. \end{aligned}$$

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Then we have

$$\begin{aligned} \mathbb{E} \sup_{t \in T} |\langle t, X \rangle|^2 = \mathbb{E} \sup_{|C^{1/2} t| \leq 1} |\langle t, X \rangle|^2 = \mathbb{E} \sup_{|C^{1/2} t| \leq 1} |\langle C^{1/2} t, C^{-1/2} X \rangle|^2 \\ = \mathbb{E} |C^{-1/2} X|^2 = n. \end{aligned}$$

Proposition (case $p = 2m$)

Let X be a random vector in $\mathbb{R}^n$ and let us take $T \subseteq \mathbb{R}^n$. Suppose m is a positive integer. Then

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Proof. Enough: for any $k, l \geq 1$ and any vectors $t_1, \ldots, t_k$ and $x_1, \ldots, x_l$ in $\mathbb{R}^n$

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From Lemma 2 the matrix $A^{\circ m} = (\langle t_i, x_j \rangle^m)$ has rank at most $N := \binom{n+m-1}{m}$. From Lemma 1 there exist vectors $\tilde{t}_1, \ldots, \tilde{t}_k$ and $\tilde{x}_1, \ldots, \tilde{x}_l$ in $\mathbb{R}^N$ such that $\langle t_i, x_j \rangle^m = \langle \tilde{t}_i, \tilde{x}_j \rangle$.

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From Lemma 2 the matrix $A^{\circ m} = (\langle t_i, x_j\rangle^m)$ has rank at most $N := \binom{n+m-1}{m}$. From Lemma 1 there exist vectors $\tilde{t}_1, \ldots, \tilde{t}_k$ and $\tilde{x}_1, \ldots, \tilde{x}_l$ in $\mathbb{R}^N$ such that $\langle t_i, x_j\rangle^m = \langle \tilde{t}_i, \tilde{x}_j\rangle$. But from Lemma 3

$$\sum_{j=1}^{l} \sup_{1 \leq i \leq k} |\langle \tilde{t}_i, \tilde{x}_j\rangle|^2 \leq N \sup_{1 \leq i \leq k} \sum_{j=1}^{l} |\langle \tilde{t}_i, \tilde{x}_j\rangle|^2.$$

$C_{n,q} \leq C_{n,p}$ if $p < q$.
Let $p_j = \sup_{i \leq k} |\langle t_i, x_j \rangle|^{q-p}$ and assume $\sum_{j=1}^{l} p_j = 1$. Let X be a random vector such that $\mathbb{P}(X = x_j) = p_j$.

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$$
\begin{aligned}
\sum_{j=1}^{l} \sup_{1\leq i\leq k} |\langle t_i, x_j\rangle|^q &= \mathbb{E} \sup_{1\leq i\leq k} |\langle t_i, X\rangle|^p \leq C_{n,p}^p \sup_{1\leq i\leq k} \mathbb{E}|\langle t_i, X\rangle|^p \\
&= C_{n,p}^p \sup_{1\leq i\leq k} \sum_{j=1}^{l} |\langle t_i, x_j\rangle|^p p_j \\
&\leq C_{n,p}^p \sup_{1\leq i\leq k} \left(\sum_{j=1}^{l} |\langle t_i, x_j\rangle|^q \right)^{p/q} \left(\sum_{j=1}^{l} p_j^{q/(q-p)} \right)^{(q-p)/q} \\
&= C_{n,p}^p \sup_{1\leq i\leq k} \left(\sum_{j=1}^{l} |\langle t_i, x_j\rangle|^q \right)^{p/q} \left(\sum_{j=1}^{l} \sup_{1\leq i\leq k} |\langle t_i, x_j\rangle|^q \right)^{(q-p)/q} .
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\begin{aligned}
\sum_{j=1}^{l} \sup_{1\le i\le k} |\langle t_i, x_j\rangle|^q &= \mathbb{E} \sup_{1\le i\le k} |\langle t_i, X\rangle|^p \le C_{n,p}^p \sup_{1\le i\le k} \mathbb{E}|\langle t_i, X\rangle|^p \\
&= C_{n,p}^p \sup_{1\le i\le k} \sum_{j=1}^{l} |\langle t_i, x_j\rangle|^p p_j \\
&\le C_{n,p}^p \sup_{1\le i\le k} \left(\sum_{j=1}^{l} |\langle t_i, x_j\rangle|^q\right)^{p/q} \left(\sum_{j=1}^{l} p_j^{q/(q-p)}\right)^{(q-p)/q} \\
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After rearranging we get $C_{n,q} \le C_{n,p}$.

Thank you!