# SL($n$) invariant valuations on convex functions

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**Examples**

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- Intrinsic Volumes: $\mu(K) = V_i(K)$

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$$\mu(K) = c_0 V_0(K) + c_n V_n(K)$$

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*with*

$$\mu_i(\lambda K) = \lambda^i \mu_i(K) \qquad \text{(}i\text{-homogeneous)}$$

*for every $K \in \mathcal{K}^n$, $\lambda \geq 0$, $i \in \{0, \dots, n\}$.*

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For $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$,

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Baryshnikov & Ghrist 2009, Baryshnikov & Ghrist 2010, Tsang 2010, Ludwig 2011, Baryshnikov & Ghrist & Lipsky 2011, Ludwig 2012, Tsang 2012, Baryshnikov & Ghrist & Wright 2013, Ober 2014, Kone 2014, Caglar & Werner 2014, Cavallina & Colesanti 2015, Ma 2016, Colesanti & Lombardi 2017, Colesanti & Lombardi & Parapatits 2018, Alesker 2019, Li & Schütt & Werner 2019 ...

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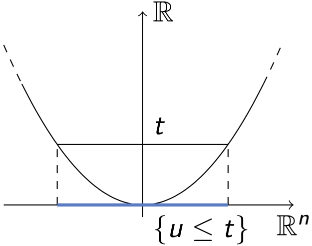


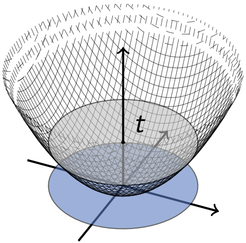

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Equipped with topology that is induced by epi-convergence.

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**Theorem (Colesanti, Ludwig, M., IMRN 2019)**

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$$\mathrm{Z}(u) = \zeta_0 \left( \min_{x \in \mathbb{R}^n} u(x) \right) + \int_{\mathrm{dom}\, u} \zeta_1(u(x)) \,\mathrm{d}x$$

*for every* $u \in \mathrm{Conv}_{\mathrm{coe}}(\mathbb{R}^n)$.

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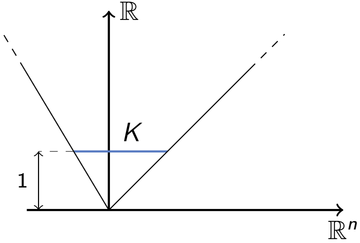


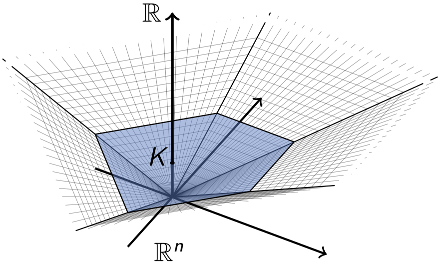

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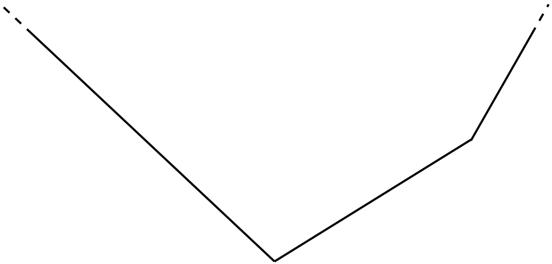

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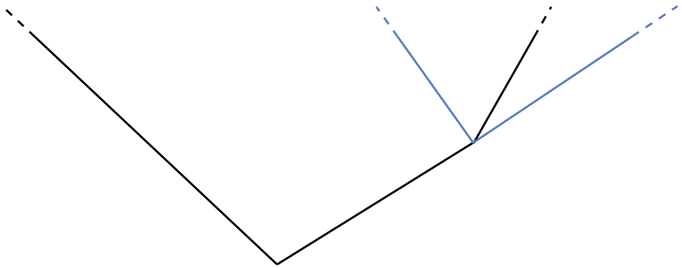

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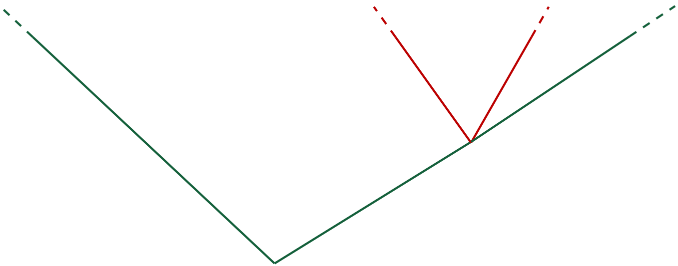

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## We are lucky!

We do not assume translation invariance here but still end up with translation invariant valuations.

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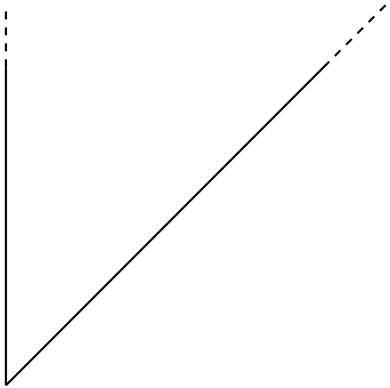

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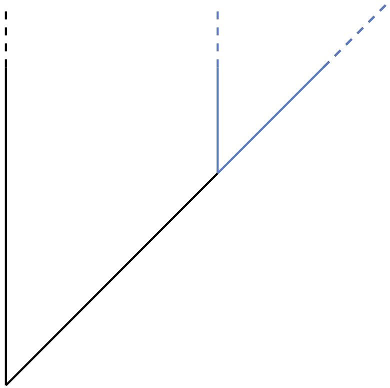

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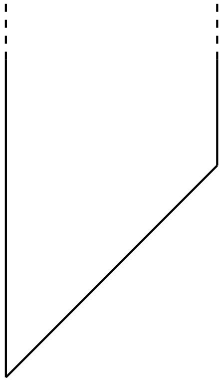

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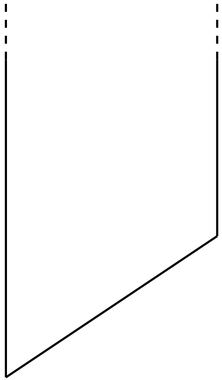

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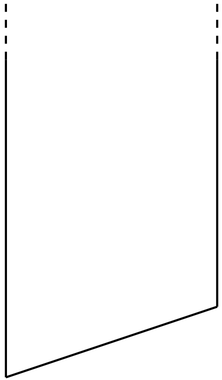

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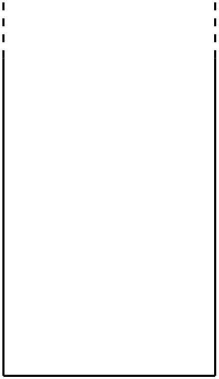

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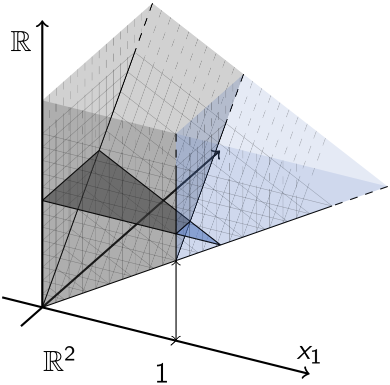

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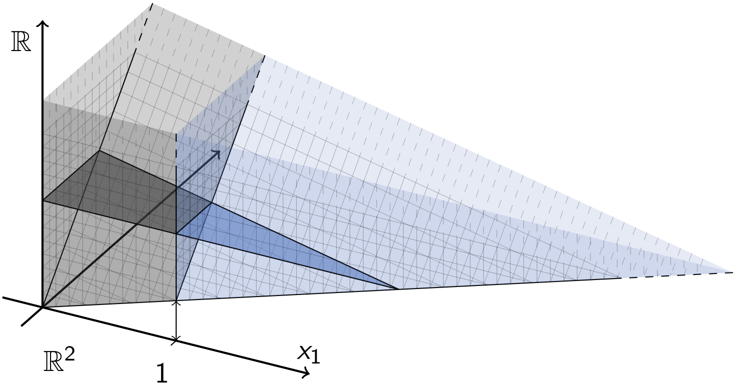

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$$\text{Z} \text{ is } i\text{-homogeneous} \iff \text{Z}\left(u(\tfrac{\cdot}{\lambda})\right) = \lambda^i \, \text{Z}(u)$$

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## Theorem (M., 2019+)

*For $n \geq 2$, a map $\mathrm{Z} : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ is a continuous, $\mathrm{SL}(n)$ and dually translation invariant valuation if and only if there exist continuous functions $\zeta_0, \zeta_1, \zeta_2 : \mathbb{R} \to [0, \infty)$ where $\int_0^\infty t^{n-1} \zeta_1(t)\,\mathrm{d}t < \infty$ and $\zeta_2(t) = 0$ for all $t \geq T$ with some $T \in \mathbb{R}$ such that*

$$\mathrm{Z}(u) = \zeta_0\left(u(0)\right) + \int_{\mathbb{R}^n} \zeta_1(u^*(x))\,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_2\big(\nabla u(x) \cdot x - u(x)\big)\,\mathrm{d}x$$

*for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$.*

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