Phase Transition Phenomena in Integral Geometry

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joint work with

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and

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Motivation

Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{$\star$}$$

When is $\boldsymbol{x}_0$ the unique solution of $(\star)$?

- Every row of $\boldsymbol{A}$ is seen as a measurement or observation that reveals information about $\boldsymbol{x}_0$.
- Motivation: applications in seismic imaging, signal processing, medical imaging, statistics and machine learning, ...

Compressed Sensing

Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

- Donoho, Candès, Romberg & Tao, Rudelson & Vershynin: Successful recovery of $\boldsymbol{x}_0$ when

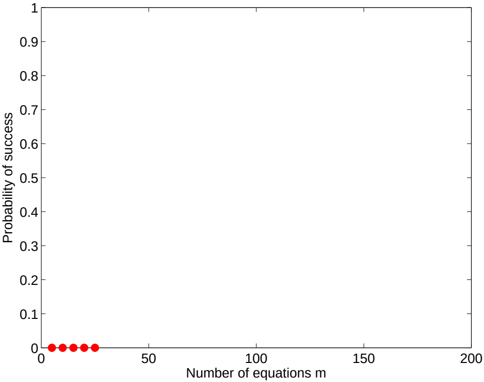
$$m \geq \mathrm{const} \cdot \log(n/m) \cdot s$$

  ("complexity" m is proportional to the "information content" s)
- Phase transitions for successful recovery were observed and precisely located by Donoho & Tanner and Stojnic

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

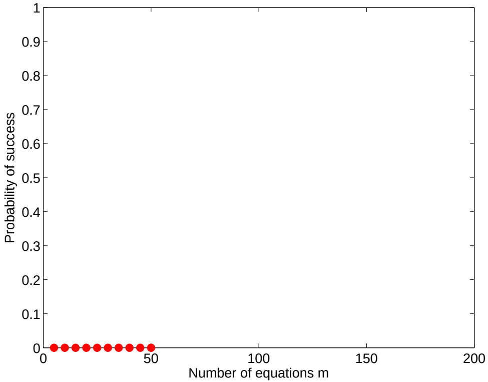


$s = 50,\ m = 25,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

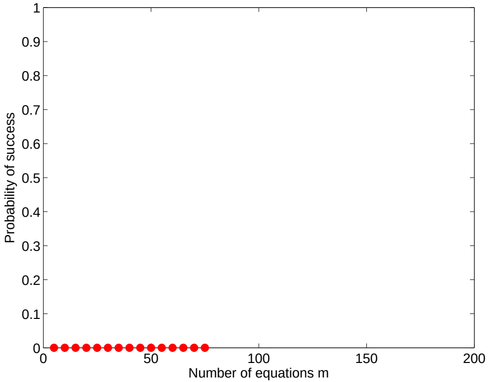


$s = 50,\ m = 50,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

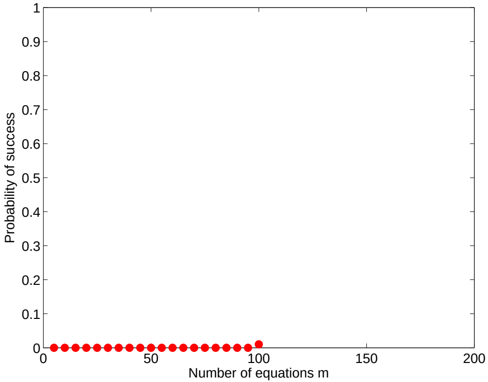


$s = 50,\ m = 75,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

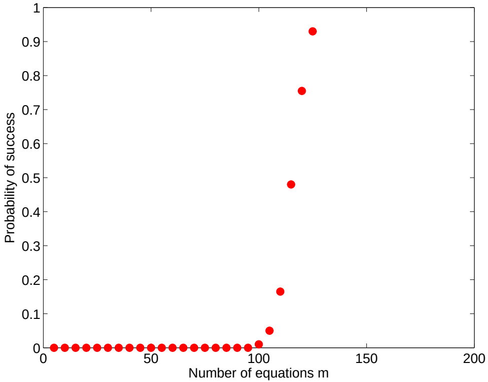


$s = 50,\ m = 100,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

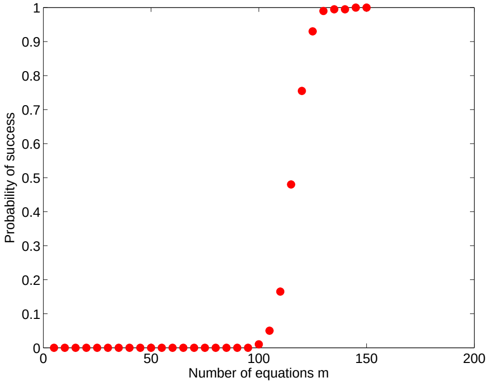


$s = 50,\ m = 125,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

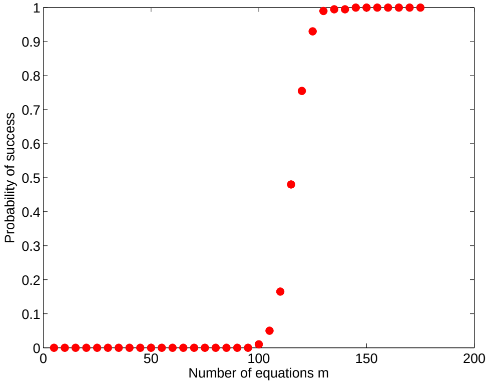


$s = 50,\ m = 150,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

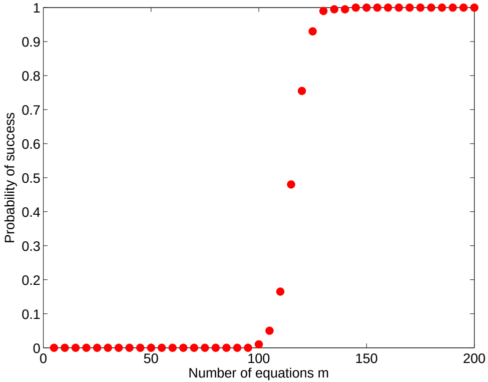


$s = 50,\ m = 175,\ n = 200$

Observed Phase Transitions

Let $\boldsymbol{x}_0$ be s-sparse, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$ for random $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ $(s < m < n)$.

$$\text{minimize} \quad \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$



$s = 50,\ m = 200,\ n = 200$

Phase Transitions for Linear Inverse Problems

Associate to a solution $\boldsymbol{x}_0$ of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and a convex problem

$$\text{minimize} \quad f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad (\star)$$

a parameter $\delta(f, \boldsymbol{x}_0)$, the statistical dimension of f at $\boldsymbol{x}_0$.

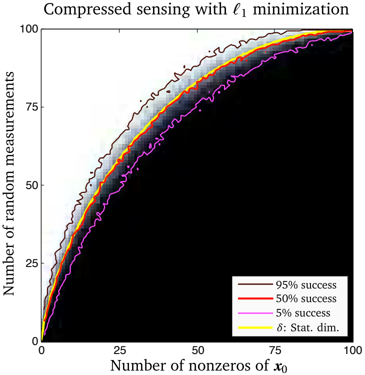
Theorem [Amelunxen, L, McCoy & Tropp, 2014]

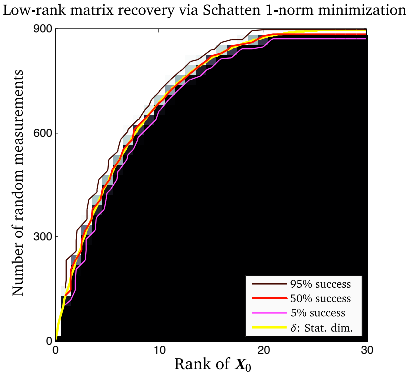
Let $\eta \in (0,1)$ and let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be a fixed vector. Suppose $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ has independent standard normal entries, and let $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}_0$. Then

$$m \geq \delta(f, \boldsymbol{x}_0) + a_\eta \sqrt{n} \quad \Longrightarrow \quad (\star) \text{ recovers } \boldsymbol{x}_0 \text{ with probability } \geq 1 - \eta;$$
$$m \leq \delta(f, \boldsymbol{x}_0) - a_\eta \sqrt{n} \quad \Longrightarrow \quad (\star) \text{ recovers } \boldsymbol{x}_0 \text{ with probability } \leq \eta.$$

where $a_\eta := 4\sqrt{\log(4/\eta)}$.

Phase Transitions for Linear Inverse Problems





From Optimization to Geometry

The problem

$$\text{minimize} \quad f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

has $\boldsymbol{x}_0$ as unique solution if and only if the optimality condition

$$\ker \boldsymbol{A} \cap \mathcal{D}(f, \boldsymbol{x}_0) = \{0\}$$

is satisfied, where

$$\mathcal{D}(f, \boldsymbol{x}_0) := \bigcup_{\tau > 0} \left\{ \boldsymbol{y} \in \mathbb{R}^n \colon f(\boldsymbol{x}_0 + \tau \boldsymbol{y}) \le f(\boldsymbol{x}_0) \right\}$$

is the convex descent cone of f at $\boldsymbol{x}_0$.

- $\boldsymbol{A}$ Gaussian $\Rightarrow \ker \boldsymbol{A}$ uniform in Grassmannian

The Mathematical Problem

Given a closed convex cone $C \subset \mathbb{R}^n$ and a random linear subspace L with $\dim L = k$, find:

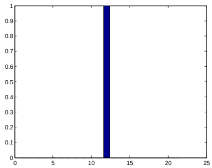
$$\mathbb{P}\{C \cap L \neq \{\mathbf{0}\}\} := \nu_k(\{L \in \mathrm{Gr}(k,n) : C \cap L \neq \{\mathbf{0}\}\}),$$

where ν_k is the normalized Haar measure on the Grassmannian.

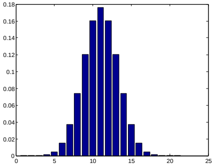
- Bounds can be derived from Gordon's escape through the mesh argument;
- Exact formulas for probability of intersection are based on the Crofton Formula from (spherical) integral geometry.

Spherical Intrinsic Volumes

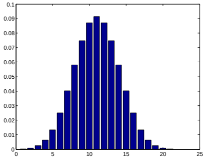
$v_0(C), \dots, v_n(C)$: spherical/conic intrinsic volumes of C.



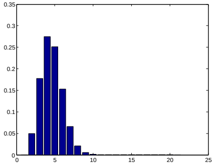
$v_k(L)$, $\dim L = k$



$v_k(\mathbb{R}^n_{\geq 0})$



$v_k(\mathrm{Circ}(n, \pi/4))$



$v_k(\{\boldsymbol{x} \colon x_1 \leq \cdots \leq x_n\})$

The Crofton and Kinematic Formula

- Kinematic Formula

$$\mathbb{P}\{C \cap \boldsymbol{Q}D \neq \{\mathbf{0}\}\} = 2\sum_i v_i(C)\left(\sum_{k \text{ odd}} v_{n-i+k}(D)\right),$$

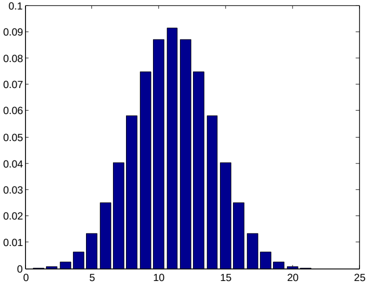
where $\boldsymbol{Q}$ uniformly distributed on $SO(n)$.

- Crofton Formula

$$\mathbb{P}\{C \cap L \neq \{\mathbf{0}\}\} = 2\sum_{k \text{ odd}} v_{m+k}(C),$$

where L uniform in $\mathrm{Gr}(n-m,n)$.

Moments



$\mathrm{Circ}(n, \pi/4)$

Associate to a cone C the discrete random variable X_C with

$$\mathbb{P}\{X_C = k\} = v_k(C),$$

and define the statistical dimension as the expectation

$$\delta(C) = \mathbb{E}[X_C] = \sum_{k=0}^{n} k v_k(C).$$

It appears that X_C might concentrate around $\delta(C)$.

Concentration of Measure

Theorem (Amelunxen-L-McCoy-Tropp 2014)

Let C be a convex cone, and X_C a discrete random variable with distribution $\mathbb{P}\{X_C = k\} = v_k(C)$. Let $\delta(C) = \mathbb{E}[X_C]$. Then

$$\mathbb{P}\{|X_C - \delta(C)| > \lambda\} \leq 4\exp\left(\frac{-\lambda^2/8}{\omega(C)+\lambda}\right) \quad \text{for } \lambda \geq 0,$$

where $\omega(C) := \min\{\delta(C), d - \delta(C)\}$.

- Refined by McCoy-Tropp 2014.
- Central Limit Theorem by Goldstein-Nourdin-Peccati 2017.

Approximate Crofton Formula

Applying the concentration result to the Crofton Formula

Corollary

Let $\eta \in (0,1)$ and assume L uniformly distributed in $\mathrm{Gr}(n-m,n)$. Then

$$\begin{aligned} \delta(C) \leq m - a_\eta \sqrt{n} \quad &\Longrightarrow \quad \mathbb{P}\big\{C \cap L = \{\mathbf{0}\}\big\} \geq 1-\eta; \\ \delta(C) \geq m + a_\eta \sqrt{n} \quad &\Longrightarrow \quad \mathbb{P}\big\{C \cap L = \{\mathbf{0}\}\big\} \leq \eta, \end{aligned}$$

where $a_\eta := 4\sqrt{\log(4/\eta)}$.

- If $C = \mathcal{D}(f, \boldsymbol{x}_0)$, define $\delta(f, \boldsymbol{x}_0) = \delta(C)$.

Approximate Kinematic Formula

Applying the concentration result to the Kinematic Formula

Corollary

Let $\eta \in (0,1)$ and assume one of C, D is not a subspace. Then

$$\begin{aligned}
\delta(C) + \delta(D) \le n - a_\eta \sqrt{n} \quad &\Longrightarrow \quad \mathbb{P}\big\{ C \cap \boldsymbol{Q} D = \{\mathbf{0}\} \big\} \ge 1 - \eta; \\
\delta(C) + \delta(D) \ge n + a_\eta \sqrt{n} \quad &\Longrightarrow \quad \mathbb{P}\big\{ C \cap \boldsymbol{Q} D = \{\mathbf{0}\} \big\} \le \eta,
\end{aligned}$$

where $a_\eta := 4\sqrt{\log(4/\eta)}$.

Euclidean Integral Geometry

Do similar results hold in Euclidean space $\mathbb{R}^n$?

Steiner Formula

$$\mathrm{Vol}_n(K + \lambda \mathsf{B}_n) = \sum_{i=0}^{n} \lambda^{n-i} \kappa_{n-i} \cdot \mathrm{V}_i(K),$$

where K convex body, $\kappa_i = \mathrm{Vol}_i(\mathsf{B}_i)$.

- $\mathrm{V}_i(K)$: i-th intrinsic volume;
- Wills functional $\mathrm{W}(K) = \mathrm{V}_0(K) + \mathrm{V}_1(K) + \cdots + \mathrm{V}_n(K)$.

Crofton and Kinematic Formula

Different normalization (Nijenhuis, ...)

$$\overline{\mathrm{V}}_i(K) := \frac{\omega_{n+1}}{\omega_{i+1}}\,\mathrm{V}_i(K), \qquad \overline{\mathrm{W}}(K) := \sum_{i=0}^{n} \overline{\mathrm{V}}_i(K),$$

where ω_k is the surface area of a k-dimensional unit ball.

- Crofton Formula

$$\int_{\mathrm{Af}(n-i,n)} \overline{\mathrm{W}}(K \cap E)\ \mu_{n-i}(\mathrm{d}E) = \sum_{k=i}^{n} \overline{\mathrm{V}}_k(K).$$

Crofton and Kinematic Formula

Different normalization (Nijenhuis, ...)

$$\overline{\mathrm{V}}_i(K) := \frac{\omega_{n+1}}{\omega_{i+1}}\, \mathrm{V}_i(K), \qquad \overline{\mathrm{W}}(K) := \sum_{i=0}^{n} \overline{\mathrm{V}}_i(K),$$

where ω_k is the surface area of a k-dimensional unit ball.

- Crofton Formula

$$\int_{\mathrm{Af}(n-i,n)} \frac{\overline{\mathrm{W}}(K \cap E)}{\overline{\mathrm{W}}(K)}\; \mu_{n-i}(\mathrm{d}E) = \sum_{k=i}^{n} \frac{\overline{\mathrm{V}}_k(K)}{\overline{\mathrm{W}}(K)}.$$

Crofton and Kinematic Formula

Different normalization (Nijenhuis, ...)

$$\overline{\mathrm{V}}_i(K) := \frac{\omega_{n+1}}{\omega_{i+1}}\,\mathrm{V}_i(K), \qquad \overline{\mathrm{W}}(K) := \sum_{i=0}^{n} \overline{\mathrm{V}}_i(K),$$

where ω_k is the surface area of a k-dimensional unit ball.

- Kinematic Formula

$$\int_{G_n} \overline{\mathrm{W}}(K \cap gM)\; \mu(\,\mathrm{d}g) = \sum_{j=0}^{n} \overline{\mathrm{V}}_j(K) \left(\sum_{k=n-j}^{n} \overline{\mathrm{V}}_k(M) \right).$$

Crofton and Kinematic Formula

Different normalization (Nijenhuis, ...)

$$\overline{\mathrm{V}}_i(K) := \frac{\omega_{n+1}}{\omega_{i+1}}\,\mathrm{V}_i(K), \qquad \overline{\mathrm{W}}(K) := \sum_{i=0}^{n} \overline{\mathrm{V}}_i(K),$$

where ω_k is the surface area of a k-dimensional unit ball.

- Kinematic Formula

$$\int_{G_n} \frac{\overline{\mathrm{W}}(K \cap gM)}{\overline{\mathrm{W}}(K)\,\overline{\mathrm{W}}(M)}\, \mu(\,\mathrm{d}g) = \sum_{j=0}^{n} \frac{\overline{\mathrm{V}}_j(K)}{\overline{\mathrm{W}}(K)} \left(\sum_{k=n-j}^{n} \frac{\overline{\mathrm{V}}_k(M)}{\overline{\mathrm{W}}(K)} \right).$$

Concentration of Intrinsic Volumes

Define the random variable I_K as

$$\mathbb{P}\{I_K = n - i\} = \frac{\mathrm{V}_i(K)}{\mathrm{W}(K)}, \quad \delta(K) := \mathbb{E}[I_K].$$

Theorem (L-Tropp, 2019)

$\mathrm{Var}[I_K] \le 2 \cdot \delta(K)$, and for $t \ge 0$,

$$\mathbb{P}\{|I_K - \delta(K)| \ge t\} \le 2\exp\left(\frac{-t^2/2}{2(\delta(K) + t/3)}\right)$$

- Similar results for $\overline{\mathrm{V}}_i(K)$ and other normalizations.

Approximate Integral Geometry

Theorem (Approximate Crofton Formula)

Let $\eta \in (0,1)$. Then

$$\int_{\mathrm{Af}(n-i,n)} \frac{\overline{\mathrm{W}}(K \cap E)}{\overline{\mathrm{W}}(K)} \; \mu_{n-i}(\mathrm{d}E) \begin{cases} \leq \eta & \bar{\delta}(K) \geq n - i + a_\eta \sqrt{\bar{\delta}(K)} \\ \geq 1 - \eta & \bar{\delta}(K) \leq n - i - a_\eta \sqrt{\bar{\delta}(K)}, \end{cases}$$

where $a_\eta := \sqrt{5 \log(\eta^{-1})}$.

Similar approximate version of Projection Formula.

Approximate Integral Geometry

Approximate Kinematic Formula

Let $\eta \in (0,1)$. Then

$$\int_{G_n} \frac{\overline{\mathrm{W}}(K \cap gM)}{\overline{\mathrm{W}}(K)\,\overline{\mathrm{W}}(M)}\ \mu(\mathrm{d}g) \begin{cases} \leq \eta & \overline{\delta}(K) + \overline{\delta}(M) \geq n + a_\eta \sqrt{\overline{\delta}(K)} \\ \geq 1 - \eta & \overline{\delta}(K) + \overline{\delta}(M) \leq n - a_\eta \sqrt{\overline{\delta}(K)}, \end{cases}$$

where $a_\eta := \sqrt{5 \log(\eta^{-1})}$.

Similar approximate version of Rotation Mean Formula.

Proof of Concentration

Let Y be any random variable.

- Moment generating function (mgf) $m_Y(\theta) := \mathbb{E}\, \mathrm{e}^{\theta Y}$;
- Cumulant generating function (cgf) $\xi_Y(\theta) = \log m_Y(\theta)$.

Cramér-Chernoff

$$\mathbb{P}\{Y \geq t\} \leq \exp\left(\inf_{\theta > 0} [\xi_Y(\theta) - \theta t]\right);$$

$$\mathbb{P}\{Y \leq -t\} \leq \exp\left(\inf_{\theta < 0} [\xi_Y(\theta) + \theta t]\right).$$

- Goal: Find bound on $\xi_Y(\theta)$ when $Y = I_K - \delta(K)$.

Proof Outline

Bound on $\xi_Y(\theta)$ can be deduced from differential inequality:

Proposition

$$\xi''_{I_K}(\theta) \le 2\xi'_{I_K}(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

- Define log-concave probability density

$$\mu_\theta(\boldsymbol{x}) := \frac{1}{\mathrm{W}(K)} \cdot \frac{1}{m_{I_K}(\theta)} \cdot \mathrm{e}^{-V_\theta(\boldsymbol{x})} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n \text{ and } \theta \in \mathbb{R}.$$

  where $V_\theta(\boldsymbol{x}) := \pi \mathrm{e}^{-2\theta} \operatorname{dist}_K^2(\boldsymbol{x})$.
- Use integral form of Steiner Formula to derive the identities

$$\xi'_{I_K}(\theta) = \mathbb{E}_\theta[2V_\theta], \qquad \xi''_{I_K}(\theta) = \mathrm{Var}_\theta[2V_\theta] - 2\xi'_{I_K}(\theta),$$

  where $\mathbb{E}_\theta$ and Var_θ are mean and variance with respect to μ_θ;

Proof Outline

- Final step: bound the variance term in

$$\xi''_{I_K}(\theta) = \mathrm{Var}_\theta[2V_\theta] - 2\xi'_{I_K}(\theta).$$

- Use Brascamp-Lieb variance inequality

$$\mathrm{Var}_\theta[f] \leq \int_{\mathbb{R}^n} \left\langle (\nabla^2 V_\theta)^{-1} \nabla f,\ \nabla f \right\rangle \mu_\theta(\,\mathrm{d}\boldsymbol{x})$$

to bound

$$\mathrm{Var}[2V_\theta] \leq 4\,\mathbb{E}[2V_\theta] = 4\xi'_{I_K}(\theta).$$

- Concentration of normalized intrinsic volumes along the lines, but uses other (concave) densities and variance bounds (Nguyen, 2014).

For more details:

D. Amelunxen, M. Lotz, M. B. McCoy, and J. A. Tropp
Living on the edge: phase transitions in convex programs with random data.
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From Steiner formulas for cones to concentration of intrinsic volumes.
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Gaussian Phase Transitions and Conic Intrinsic Volumes: Steining the Steiner Formula.
The Annals of Applied Probability, 2017

M. Lotz, M. B. McCoy, I. Nourdin, G. Peccati, and J. A. Tropp
Concentration of the Intrinsic Volumes of a Convex Body.
Geometric Aspects of Functional Analaysis - Israel Seminar (GAFA), 2017-2019 (to appear)

M. Lotz and J. A. Tropp
Phase Transitions in Integral Geometry.
In preparation, 2019 (?).

Thank You!