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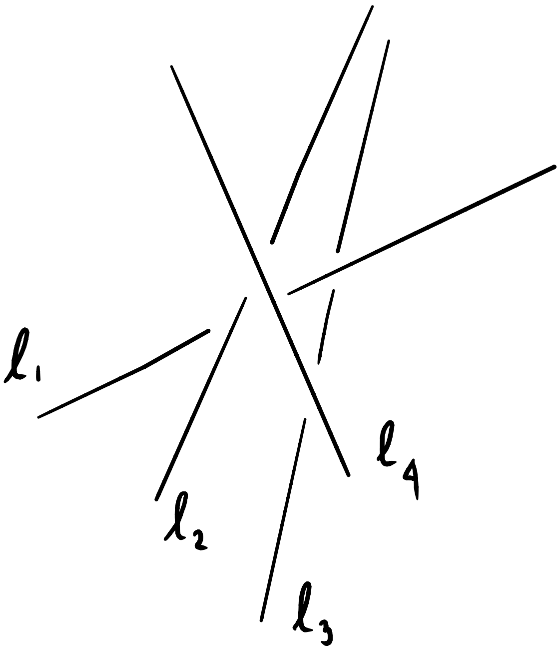
PROBABILISTIC SCHUBERT CALCULUS

(joint work with P. BÜRGISSER)

QUESTION: how many lines intersect 4 generic lines in 3-dim space?



$$\#_{\mathbb{C}} = 2$$

$$\#_{\mathbb{R}} = 0, 2$$

$$\equiv 2 \bmod 2$$

PROOF OVER $\mathbb{C}$:

$$\Omega(\ell) = \{\text{lines intersecting } \ell\} \subseteq \mathbb{G}(1,3)$$

SCHUBERT VARIETY

GRASSMANNIAN OF LINES IN 3-dim PROJECTIVE SPACE

$$[\Omega(\ell_1) \cap \Omega(\ell_2) \cap \Omega(\ell_3) \cap \Omega(\ell_4)] \in H_0(\mathbb{G}(1,3); \mathbb{Z})$$

$$\updownarrow \qquad\qquad\qquad\qquad \cong$$

$$[\,\square \smile \square \smile \square \smile \square\,] \in H^{2\cdot 4}(\mathbb{G}(1,3); \mathbb{Z})$$

SCHUBERT CALCULUS ("how to compute in $H^*(\mathbb{G})$)

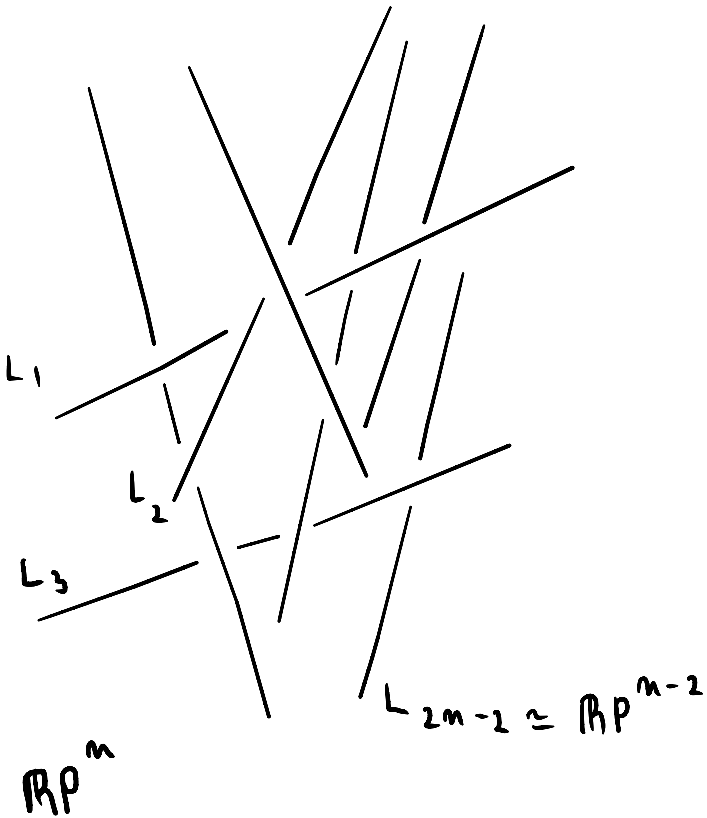
$$\#_{\mathbb{C}} = \square^4 = 2$$

$$\{\,\cdot\,, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,2}\}$$

YOUNG DIAGRAMS, LABEL A NATURAL BASIS FOR $H^*(\mathbb{G})$

$$\square^4 = \left(\ydiagram{2} + \ydiagram{1,1}\right)\cdot \square^2 = \left(\ydiagram{2,1} + \ydiagram{2,1}\right)\cdot \square = 2\,\ydiagram{2,2}$$

$$L_1, L_2, \dots, L_{2n-2} \simeq \mathbb{RP}^{n-2} \subset \mathbb{RP}^n$$



$\#\{$ lines intersecting all the $L_i \}$

$$\#_{\mathbb{C}} = \frac{1}{n}\binom{2n-2}{n-1}$$

$$\sim \frac{1}{\pi^{1/2}} \cdot \frac{1}{n^{3/2}} \cdot 4^{n-1}$$

$\#_{\mathbb{R}} =$ too many possibilities!

A PROBABILISTIC APPROACH

QUESTION: how many lines intersect 4 random lines in 3-dim space?

$\ell_1, \ell_2, \ell_3, \ell_4$ i.i.d. from $\mathbb{G}(1,3) = G(2,4)$

$$\ell_i = P\left(\operatorname{span}\begin{bmatrix} \xi_{11} & \xi_{21} \\ \xi_{12} & \xi_{22} \\ \xi_{13} & \xi_{23} \\ \xi_{14} & \xi_{24} \end{bmatrix}\right) \qquad \xi_{ij} \sim N(0,1)$$

$$\mathbb{E}\,\#\,\Omega(\ell_1)\cap\Omega(\ell_2)\cap\Omega(\ell_3)\cap\Omega(\ell_4) =$$

$$\mathbb{E}\,\#\,g_1\Omega \cap g_2\Omega \cap g_3\Omega \cap g_4\Omega =$$

$\underline{g} \in O(4)^4$ $\qquad \Omega = \Omega(\ell_0)$

$$=: \delta_{1,3}$$

lines $\quad$ in $\mathbb{RP}^3$

$\left(\text{"}\delta^{\mathbb{C}}_{1,3} = 2\text{"}\right)$

4

$\delta_{1,3}$ = Kinematic integral in $\mathbb{G}(1,3)$

$$= \underset{g_1,\dots,g_4 \in O(4)}{\mathbb{E}} \# \, g_1\Omega_1 \cap \dots \cap g_4\Omega_4$$

= no exact formula in general but ...

$\mathbb{G}(1,3) = G(2,4) = O(4) / O(2) \times O(2)$ (stabilizer is too small!)

= ... exact formula in this case! (exact kinematic formulas in $G(k,n)$, if we intersect SCHUBERT!)

$$= \left(\prod_{i=1}^{4} \mathrm{vol}(\Omega_i) \right) \cdot \text{const}$$

ex $\underset{\underline{g} \in O(4)^4}{\mathbb{E}} \# \, g_1A_1 \cap g_2A_2 \cap g_3A_3 \cap g_4A_4 = \mathrm{vol}(A_1) \cdots \mathrm{vol}(A_4) \cdot \text{const}$ $\quad A_i \hookrightarrow S^4$

CONVEX BODY IN $\mathbb{R}^{2\times 2}$

DEPENDS ONLY ON SING. VALUES

$$h_C(M) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(x^2\sigma_1(M)^2 + y^2\sigma_2(M)^2\right)^{1/2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

$$\delta_{1,3} = \frac{\mathrm{vol}(G(1,3))}{2^4}\cdot 4!\,\mathrm{vol}(C(1,3))$$

$4!\,\mathrm{vol}(C(1,3)) = V(C(1,3),\dots,C(1,3))$

cohomological interpretation!

MORE GENERALLY:

$$\delta_{k,n} = \mathbb{E}\#(\cdots) = \frac{\mathrm{vol}(G(k,n))}{2^{(k+1)(n-k)}}\cdot\big((k+1)(n-k)\big)!\,\mathrm{vol}(C(k,n))$$

$L_1,\dots,L_{(k+1)(n-k)} \simeq \mathbb{RP}^{n-k-1} \subset \mathbb{RP}^n$

$\mathbb{E}\#\{k\text{-flats intersecting all of the } L_i\}$

$\delta_{k,n}$ = universal constant governing questions in random enumerative geometry.

ex. $\mathbb{E}\#$ lines meeting $c_1,\dots,c_4$ $= \delta_{1,3}\cdot\prod \mathrm{vol}(c_i)$

Convex bodies allow to perform asymptotics:

$$\delta_{1,n} = \frac{8}{3\pi^{5/2}} \frac{1}{n^{1/2}} \left(\frac{\pi^2}{4}\right)^n \left(1+O(n^{-1})\right)$$

$$\delta^{\mathbb{C}}_{1,n} = \frac{1}{4\pi^{1/2}} \cdot \frac{1}{n^{3/2}} \cdot 4^n \left(1+O(n^{-1})\right)$$ COMBINATORICS

(L. MATHIS) $$\delta_{k,n} = a_k \cdot n^{-\frac{k(k+1)}{4}} \cdot b_k^n \left(1+O(n^{-1})\right)$$

$$\delta^{\mathbb{C}}_{k,n} = a^{\mathbb{C}}_k \, n^{-\frac{k(k+2)}{2}} \left(b^{\mathbb{C}}_k\right)^n \left(1+O(n^{-1})\right)$$

($\delta_{k,n}$ is a "period" in the sense of KONTSEVICH-ZAGIER)

MESSAGE: in general there are no exact kinematic formula in homogeneous spaces, but if we restrict the class of submanifolds we intersect (e.g. schubert varieties) such formula exist!

The structure constants involve operations on certain convex bodies associated to the submanifolds we intersect!

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