On the maximal perimeter of sections of the cube

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Keith Ball showed that the hyperplane section of the unit n-cube B_∞^n perpendicular to $a_{max} := \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0)$ has maximal $(n-1)$-dimensional volume among all hyperplane sections, i.e. for any $a \in S^{n-1} \subset \mathbb{R}^n$

$$vol_{n-1}(B_\infty^n \cap a^\perp) \le vol_{n-1}(B_\infty^n \cap a_{max}^\perp) = \sqrt{2},$$

where $a^\perp$ is the central hyperplane orthogonal to a. Oleszkiewicz and Pełczyński proved a complex analogue of this result, with the same hyperplane $a_{max}^\perp$.

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Pełczyński asked whether the same hyperplane section is also maximal for intersections with the *boundary* of the n-cube, i.e. whether for all $a \in S^{n-1} \subset \mathbb{R}^n$

$$vol_{n-2}(\partial B_\infty^n \cap a^\perp) \le vol_{n-2}(\partial B_\infty^n \cap a_{max}^\perp) = 2((n-2)\sqrt{2} + 1).$$

He proved it for $n = 3$ when $vol_1(\partial B_\infty^3 \cap a^\perp)$ is the *perimeter* of the quadrangle or hexagon of intersection.

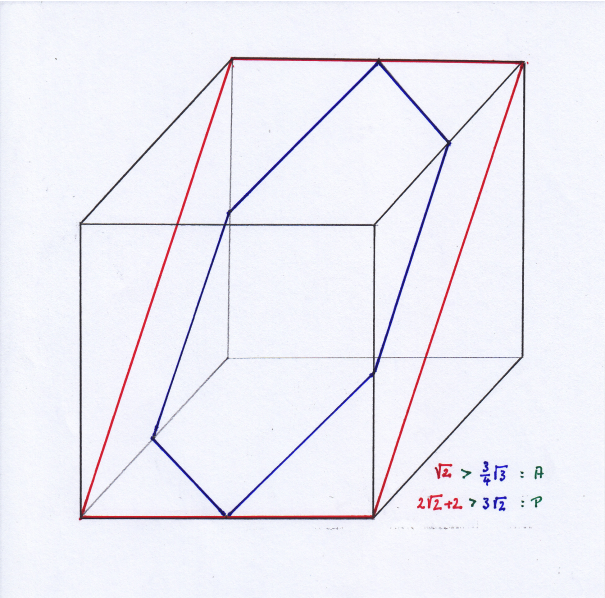

Abbildung: Cubic sections

Notation.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\alpha = \frac{1}{2}$ for $\mathbb{K} = \mathbb{R}$ and $\alpha = \frac{1}{\sqrt{\pi}}$ for $\mathbb{K} = \mathbb{C}$.
Let $|| \cdot ||_\infty$ and $| \cdot |$ be the maximum and the Euclidean norm on $\mathbb{K}^n$ and

$$B_\infty^n := \{x \in \mathbb{K}^n \mid ||x||_\infty \leq \alpha\}$$

be the n-cube of volume 1 in $\mathbb{K}^n$. For $\mathbb{K} = \mathbb{C}$, identify $\mathbb{C}^k = \mathbb{R}^{2k}$ for volume calculations. For $a \in \mathbb{K}^n$ with $|a| = 1$ and $t \in \mathbb{K}$, the *parallel section function* A is defined by

$$A_{n-1}(a, t) := vol_{l(n-1)}(B_\infty^n \cap (a^\perp + \alpha t a)),$$

with $l = 1$ if $\mathbb{K} = \mathbb{R}$ and $l = 2$ if $\mathbb{K} = \mathbb{C}$. Put $A_{n-1}(a) = A_{n-1}(a, 0)$.

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$$A_{n-1}(a) \leq A_{n-1}(a_{max}) = (\sqrt{2})^l.$$

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For $a \in \mathbb{K}^n$ with $|a| = 1$, define the *perimeter* of the cubic section by $a^\perp$ as

$$P_{n-2}(a) := vol_{l(n-2)}(\partial B_\infty^n \cap a^\perp), \; l \text{ as before.}$$

The answer to Pełczyński's problem for

$$P_{n-2}(a) := vol_{l(n-2)}(\partial B_\infty^n \cap a^\perp)$$

is affirmative. This is a joint result with A. Koldobsky:

Theorem 1

Let $n \geq 3$ *and* $a_{max} := \frac{1}{\sqrt{2}}(1, 1, 0, \cdots, 0) \in \mathbb{K}^n$. *Then for any* $a \in \mathbb{K}^n$ *with* $|a| = 1$ *we have*

$$P_{n-2}(a) \leq P_{n-2}(a_{max}), \tag{1}$$

We have

$$P_{n-2}(a_{max}) = 2((n-2)\sqrt{2} + 1) \quad , \quad \mathbb{K} = \mathbb{R}$$

and

$$P_{n-2}(a_{max}) = 2\pi((n-2)2 + 1) \quad , \quad \mathbb{K} = \mathbb{C}.$$

As a consequence of Theorem 1 we find a counterexample to a surface area version of the Busemann-Petty for large dimensions (König, Koldobsky):

Theorem 2

For each $n \geq 14$, there exist origin-symmetric convex bodies K, L in $\mathbb{R}^n$ such that for all $a \in S^{n-1}$

$$vol_{n-2}(\partial K \cap a^{\perp}) \leq vol_{n-2}(\partial L \cap a^{\perp})$$

but

$$vol_{n-1}(\partial K) > vol_{n-1}(\partial L).$$

Example. Let $K = B_{\infty}^n$ be the unit cube in $\mathbb{R}^n$. Let L be the Euclidean ball of radius r in $\mathbb{R}^n$ so that the perimeters of hyperplane sections of L are all equal to the maximal perimeter of sections of K.

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$$vol_{n-2}(\partial K \cap a^\perp) \leq vol_{n-2}(\partial K \cap a_{max}^\perp) = 2((n-2)\sqrt{2}+1)$$
$$= vol_{n-2}(rS^{n-2}) = r^{n-2} \ \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \ ,$$

$$r = \frac{[((n-2)\sqrt{2}+1)\Gamma(\frac{n-1}{2})]^{\frac{1}{n-2}}}{\pi^{(n-1)/(2(n-2))}}.$$

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The opposite inequality for the surface areas of K and L happens when

$$vol_{n-1}(\partial B_\infty^n) = 2n > vol_{n-1}(rS^{n-1}) = r^{n-1}\ \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\ ,$$

$$1 > \frac{\pi^{n/2}}{n\Gamma(\frac{n}{2})} r^{n-1} = \frac{1}{n\Gamma(\frac{n}{2})} \frac{[((n-2)\sqrt{2}+1)\Gamma(\frac{n-1}{2})]^{\frac{n-1}{n-2}}}{\pi^{1/(2(n-2))}} =: BP(n)\ .$$

Then BP is decreasing in n, with $BP(x_0) = 1$ for $x_0 \simeq 13.70$, so $BP(n) < 1$ for all $n \geq 14$.

In the complex case, a similar counterexamples exists for all $n \geq 11$.

For $a \in \mathbb{K}^n$ with $|a| = 1$ let $a^\star$ denote the non-increasing rearrangement of the sequence $(|a_k|)_{k=1}^n$. Then

$$A_{n-1}(a,t) = A_{n-1}(a^\star, |t|) \quad , \quad P_{n-2}(a) = P_{n-2}(a^\star).$$

Thus assume that $a = (a_k)_{k=1}^n$ satisfies $a_1 \geq \cdots \geq a_n \geq 0$, $|a| = 1$ and $t \geq 0$.

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$$A_{n-1}(a,t) = \frac{1}{2} \int_0^\infty \prod_{k=1}^n j_1(a_k s)\, J_0(ts)\, s\, ds \quad , \quad \mathbb{K} = \mathbb{C}, \tag{3}$$

where $j_1(t) = 2\frac{J_1(t)}{t}$ and J_ν denote the Bessel functions of index ν. If $a_k = 0$, $\frac{\sin(a_k s)}{a_k s}$ and $j_1(a_k s)$ have to be read as 1 in formulas (2) and (3).

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Formula (2) is due to Pólya 1913 and was used by Ball in his proof. Both formulas can be shown by taking the Fourier transform of $A_{n-1}(a,\cdot)$, using Fubini's theorem and taking the inverse Fourier transform. The $\frac{\sin t}{t}$ and $j_1(t)$ functions occur as Fourier transforms of the interval in $\mathbb{R}$ and the disc in $\mathbb{C} = \mathbb{R}^2$, respectively.

To prove Theorem 1, we use the following formulas for the perimeter.

Proposition 2

For any $a = (a_k)_{k=1}^n \in S^{n-1} \subset \mathbb{R}^n$

$$P_{n-2}(a) = 2 \sum_{k=1}^{n} \sqrt{1 - a_k^2} \, \frac{2}{\pi} \int_0^\infty \prod_{j=1, j \neq k}^{n} \frac{\sin(a_j s)}{a_j s} \cos(a_k s) \, ds \quad , \quad \mathbb{K} = \mathbb{R}, \qquad (4)$$

$$P_{n-2}(a) = 2\pi \sum_{k=1}^{n} (1 - a_k^2) \, \frac{1}{2} \int_0^\infty \prod_{j=1, j \neq k}^{n} j_1(a_j s) \, J_0(a_k s) \, s \, ds \quad , \quad \mathbb{K} = \mathbb{C}. \qquad (5)$$

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In Ball's result, the integral in (2) for $t = 0$ is estimated by using Hölder's inequality if $a_1 \leq \frac{1}{\sqrt{2}}$, which is natural since in the extremal case ($a_1 = a_2 = \frac{1}{\sqrt{2}}, a_j = 0, j > 3$) the integrand is non-negative. In (4) and (5) we have weighted sums of integrals where the integrands are non-positive in the extremal case. Estimating $P_{n-2}(a)$ requires further methods in addition to those of Ball. The idea is to consider the perimeter estimate as a constrained optimization problem, using Proposition 3 below.

Proof of (4). Let $\mathbb{K} = \mathbb{R}$, $a = (a_k)_{k=1}^n \in S^{n-1}$, $a_1 \geq \cdots a_n \geq 0$, $x \in \mathbb{K}^n$, $a = (a_1, \tilde{a})$, $x = (x_1, \tilde{x})$. The hyperplane $a^\perp$ intersects the boundary ∂B_∞^n in $2n$ (typically non-central) $(n-2)$-dimensional sections of an $(n-1)$-cube, namely for $x_j = \pm\frac{1}{2}$, $j = 1, \cdots, n$.

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Take $x_1 = -\frac{1}{2}$, put $a_j' := \frac{a_j}{\sqrt{1-a_1^2}}$, $j = 1, \cdots, n$, $\tilde{a}' := (a_j')_{j=2}^n$. Then $|\tilde{a}'|^2 = 1$. By (2)

$$vol_{n-2}\{ \tilde{x} \in \mathbb{R}^{n-1} \mid \langle \tilde{x}, \tilde{a} \rangle = -x_1 a_1 = \frac{1}{2} a_1 \} = A_{n-2}(\tilde{a}', a_1')$$

$$= \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a_j' r)}{a_j' r} \cos(a_1' r) dr = \sqrt{1 - a_1^2}\, \frac{2}{\pi} \int_0^\infty \prod_{j=2}^n \frac{\sin(a_j s)}{a_j s} \cos(a_1 s) ds \;.$$

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The same holds for $x_1 = +\frac{1}{2}$ and similarly for $x_j = \pm\frac{1}{2}$, so that

$$P_{n-2}(a) = 2 \sum_{k=1}^n \sqrt{1-a_k^2}\ \frac{2}{\pi} \int_0^\infty \prod_{j=1, j\neq k}^n \frac{\sin(a_j s)}{a_j s}\ \cos(a_k s)\ ds\ ,$$

which proves (4).

For any $a = (a_k)_{k=1}^n \in S^{n-1} \subset \mathbb{R}^n$ and $k \in \{1, \cdots, n\}$, define

$$D_k(a) := \left\{ \begin{array}{ll} \frac{2}{\pi} \int_0^\infty \prod_{j=1, j \neq k}^n \frac{\sin(a_j s)}{a_j s} \cos(a_k s) \, ds \quad , & \mathbb{K} = \mathbb{R} \\ \frac{1}{2} \int_0^\infty \prod_{j=1, j \neq k}^n j_1(a_j s) \; J_0(a_k s) \; s \, ds \quad , & \mathbb{K} = \mathbb{C} \end{array} \right\}$$

so that by (4) and (5)

$$P_{n-2}(a) = 2\pi^{l-1} \sum_{k=1}^n (1 - a_k^2)^{l/2} \; D_k(a). \tag{6}$$

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Proposition 3

We have

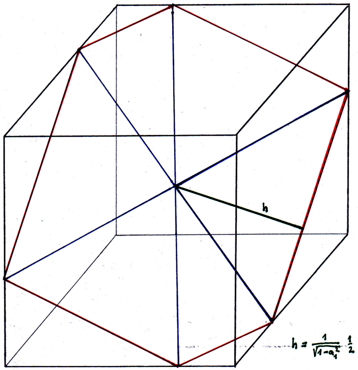
$$\sum_{k=1}^n D_k(a) = (n-1) \; A_{n-1}(a). \tag{7}$$

and for all $k \in \{1, \cdots, n\}$

$$D_k(a) \leq A_{n-1}(a).$$

Let $P(1)$ be the $(n-1)$-dimensional pyramid with vertex 0, height $h = \frac{1}{2}\frac{1}{\sqrt{1-a_1^2}}$ and base being the $(n-2)$-dimensional section with area $A_{n-2}(\tilde{a}', a_1')$. Then

$$vol_{n-1}(P(1)) = \frac{1}{n-1}\frac{1}{2}\frac{1}{\sqrt{1-a_1^2}}\ A_{n-2}(\tilde{a}', a_1') = \frac{1}{2(n-1)} D_1(a)\ .$$



Graphic illustration for $D_k(a) \leq A_{n-1}(a)$.

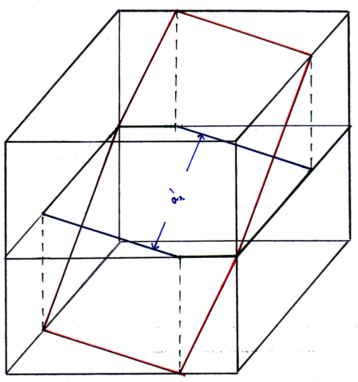


Abbildung: Scaling up one dimension

In Pełczyński's case $\mathbb{K} = \mathbb{R}$, $n = 3$, formula (4) gives explicitly

$$\frac{1}{2} P_1(a) = \left\{ \begin{array}{ll} \frac{1}{a_1}\left(\sqrt{1-a_2^2} + \sqrt{1-a_3^2}\right) & , \; a_1 \geq a_2 + a_3 \\ \sqrt{1-a_1^2}\,\frac{a_2+a_3-a_1}{2a_2a_3} + \sqrt{1-a_2^2}\,\frac{a_1+a_3-a_2}{2a_1a_3} + \sqrt{1-a_3^2}\,\frac{a_1+a_2-a_3}{2a_1a_2} & , \; a_1 < a_2 + a_3 \end{array} \right\}$$

This may be used to directly prove that

$$P_1(a) \leq 2(\sqrt{2}+1) = P_1(a_{max}).$$

Proof of Theorem 1. In the real case $\mathbb{K} = \mathbb{R}$, we have

$$D_1(a_{max}) = D_2(a_{max}) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\frac{s}{\sqrt{2}})}{\frac{s}{\sqrt{2}}} \cos(\frac{s}{\sqrt{2}}) \, ds = \frac{1}{\sqrt{2}} \frac{2}{\pi} \int_0^\infty \frac{\sin(t)}{t} \, dt = \frac{1}{\sqrt{2}},$$

$$D_j(a_{max}) = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin(\frac{s}{\sqrt{2}})}{\frac{s}{\sqrt{2}}} \right)^2 ds = \sqrt{2} \frac{2}{\pi} \int_0^\infty \left(\frac{\sin(t)}{t} \right)^2 dt = \sqrt{2} \ , \ j > 2.$$

Hence by (6)

$$P_{n-2}(a_{max}) = 2((n-2)\sqrt{2} + 1).$$

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Hence by (6)

$$P_{n-2}(a_{max}) = 2((n-2)\sqrt{2} + 1).$$

Now let $a = (a_k)_{k=1}^n \in S^{n-1}$ be arbitrary with $a_1 \geq \cdots \geq a_n \geq 0$. By (6), we get using Proposition 3

$$\frac{1}{2} P_{n-2}(a) \leq \sup\{ \sum_{k=1}^n \sqrt{1 - a_k^2}\ C_k \mid 0 \leq C_k \leq A_{n-1}(a),\ \sum_{k=1}^n C_k = (n-1)\ A_{n-1}(a) \}.$$

Since $(\sqrt{1 - a_k^2}\ )_{k=1}^n$ is increasing in k, the supremum is attained for increasing C_k and, in fact, for $C_1 = 0$, $C_2 = \cdots = C_k = A_{n-1}(a)$ so that

$$\frac{1}{2} P_{n-2}(a) \leq \sum_{k=2}^n \sqrt{1 - a_k^2}\ A_{n-1}(a).$$

$$\frac{1}{2} P_{n-2}(a) \le \sum_{k=2}^{n} \sqrt{1 - a_k^2}\ A_{n-1}(a). \tag{8}$$

Since $\phi(x) = \sqrt{1-x}$ is concave,

$$\frac{1}{n-1} \sum_{k=2}^{n} \phi(a_k^2) \le \phi\Big(\frac{1}{n-1} \sum_{k=2}^{n} a_k^2\Big) = \phi\Big(\frac{1}{n-1}(1 - a_1^2)\Big),$$

$$\frac{1}{2} P_{n-2}(a) \le (n-1)\sqrt{1 - \frac{1-a_1^2}{n-1}}\ A_{n-1}(a) \le \Big(n - 1 - \frac{1-a_1^2}{2}\Big)\ A_{n-1}(a).$$

$$\frac{1}{2}P_{n-2}(a) \le \sum_{k=2}^{n} \sqrt{1-a_k^2}\ A_{n-1}(a). \tag{8}$$

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If $a_1 \le \frac{1}{\sqrt{2}}$, we use that, by Ball's result, $A_{n-1}(a) \le \sqrt{2}$ to get

$$\frac{1}{2}P_{n-2}(a) \le \Big(n-\frac{3}{2}+\frac{a_1^2}{2}\Big)\sqrt{2} \le (n-2)\sqrt{2}+\frac{3}{4}\sqrt{2}.$$

If $a_1 > \frac{1}{\sqrt{2}}$, we use that $A_{n-1}(a) \le \frac{1}{a_1}$ and find

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However, $\frac{3}{4}\sqrt{2} \simeq 1.0607 > 1$, so that this does not prove $P_{n-2}(a) \le P_{n-2}(a_{max})$ for all $a \in S^{n-1}$. However, if a_1 satisfies $a_1 \notin \Big(\sqrt{\sqrt{2}-1}, \frac{1}{\sqrt{\sqrt{2}+\frac{1}{2}}}\Big) \simeq (0.643, 0.723)$, the above estimate yields $P_{n-2}(a) \le P_{n-2}(a_{max})$. The difficulty is that in (6) the extremals for the sum of weights and for the section function A occur for different sequences a.

In the complex case, the optimization techniques works for all a. By (6)

$$\frac{1}{2\pi} P_{n-2}(a) = \sum_{k=1}^{n} (1 - a_k^2)\ D_k(a),$$

and using Proposition 3, we have

$$\frac{1}{2\pi} P_{n-2}(a) \leq \sup\{\sum_{k=1}^{n} (1 - a_k^2)\ C_k \mid 0 \leq C_k \leq A_{n-1}(a),\ \sum_{k=1}^{n} C_k = (n-1)\ A_{n-1}(a)\}.$$

Since $(1 - a_k^2)_{k=1}^n$ is increasing in k, the sum $\sum_{k=1}^n (1 - a_k^2)\ C_k$ will be maximal for the increasing sequence $C_1 = 0,\ C_2 = \cdots = C_n = A_{n-1}(a)$. Therefore

$$\frac{1}{2\pi} P_{n-2}(a) \leq \sum_{k=2}^{n} (1 - a_k^2)\ A_{n-1}(a) = (n - 2 + a_1^2)\ A_{n-1}(a).$$

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If $a_1 \leq \frac{1}{\sqrt{2}}$, we use $A_{n-1}(a) \leq A_{n-1}(a_{max}) = 2$, so that

$$\frac{1}{2\pi} P_{n-2}(a) \leq (n - \frac{3}{2})\ 2 = \frac{1}{2\pi} P_{n-2}(a_{max}).$$

If $a_1 > \frac{1}{\sqrt{2}}$, we use that $A_{n-1}(a) \leq \frac{1}{a_1^2}$, so that

$$\frac{1}{2\pi} P_{n-2}(a) \leq (n - 2 + a_1^2) \frac{1}{a_1^2} = \frac{n-2}{a_1^2} + 1 \leq (n-2)\ 2 + 1 = \frac{1}{2\pi} P_{n-2}(a_{max}).$$

This proves Theorem 1 in the case of complex scalars.

In the remaining case of real scalars when $a_1 \in (0.643, 0.723)$ we have to improve the general estimate (8)

$$\frac{1}{2} P_{n-2}(a) \leq \sum_{k=2}^{n} \sqrt{1 - a_k^2} \, A_{n-1}(a)$$

by strengthening Ball's estimate $A_{n-1}(a) \leq \min(\sqrt{2}, \frac{1}{a_1})$ for a_1 close to $\sqrt{2}$.

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Ball's estimate for $A_{n-1}(a)$ relies on the non-trivial estimate $f(p) \leq f(2) = 1$ for the function

$$f(p) := \sqrt{\frac{p}{2}} \; \frac{2}{\pi} \int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \, dt \; ,$$

since then for all $0 < a_n \leq \cdots \leq a_1 \leq \frac{1}{\sqrt{2}}$ with $\sum_{k=1}^{n} a_k^2 = 1$ we get by using Hölder's inequality with $p_k := a_k^{-2} \geq 2$

$$\begin{aligned} A_{n-1}(a) &\leq \prod_{k=1}^{n} \left(\frac{2}{\pi} \int_0^\infty \left| \frac{\sin(a_k s)}{a_k s} \right|^{a_k^{-2}} \, ds \right)^{a_k^2} \\ &= \left(\prod_{k=1}^{n} f(a_k^{-2}) \right)^{a_k^2} \sqrt{2} \leq \sqrt{2}. \end{aligned}$$

However, for $p > 2$, $f(p) < f(2) = 1$. More precisely:

Proposition 4

Define $f : (1, \infty) \to \mathbb{R}_+$ *by*

$$f(p) := \sqrt{\frac{p}{2}}\,\frac{2}{\pi} \int_0^\infty \left| \frac{\sin(t)}{t} \right|^p \, dt.$$

Then

(a) $\lim_{p \to \infty} f(p) = \sqrt{\frac{3}{\pi}}$ *and for all* $p \geq \frac{9}{4}$, $f(p) \leq \sqrt{\frac{3}{\pi}}$.

(b) $f(\sqrt{2} + \frac{1}{2}) < \frac{51}{50}$.

(c) $f|_{[\sqrt{2}+\frac{1}{2}, \frac{9}{4}]}$ *is decreasing and convex.*

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(c) $f|_{[\sqrt{2}+\frac{1}{2}, \frac{9}{4}]}$ *is decreasing and convex.*

Using the convexity of $f|_{[\sqrt{2}+\frac{1}{2}, \frac{9}{4}]}$ and the estimates for $f(p)$ for $p = \sqrt{2} + 1/2 \simeq 1.914$ and $p = \frac{9}{4} = 2.25$, we strengthen the general estimate (8) for sequences with a_1 close to $\frac{1}{\sqrt{2}}$ in order to prove Theorem 1 in these cases. This works since $\sqrt{\frac{3}{\pi}} = \lim_{p\to\infty} f(p) < f(2) = 1$, i.e. f has strictly smaller values near ∞ than at 2.

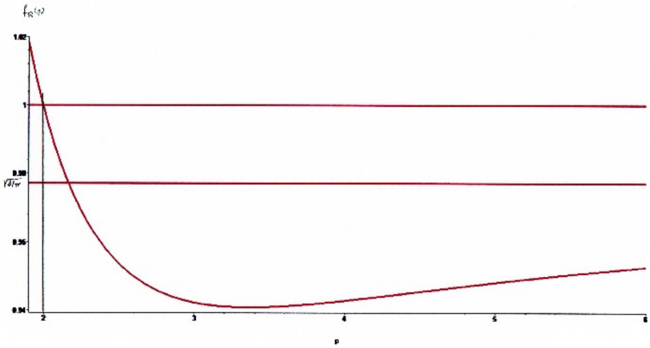

Abbildung: Ball's function

One finds e.g. in the case $\frac{2}{3} \le a_2 \le a_1 \le \frac{1}{\sqrt{2}}$, $2 \le a_1^{-2} \le a_2^{-2} \le \frac{9}{4}$ by using Hölder's inequality, the general arithmetic-geometric mean inequality and the definition of f that

$$A_{n-1}(a) \le [\ (1 - a_1^2 - a_2^2)\sqrt{\frac{3}{\pi}} + a_2^2 f(a_2^{-2}) + a_1^2 f(a_1^{-2})\ ]\sqrt{2}.$$

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Using the convexity of f to interpolate between the values $f(2) = 1$ and $f(\frac{9}{4}) < \sqrt{3/\pi}$, one concludes for $k = 1, 2$

$$a_k^2 f(a_k^{-2}) \le a_k^2(\lambda_k f(2) + (1-\lambda_k) f(\frac{9}{4})) \le (9a_k^2 - 4) + (4 - 8a_k^2)\sqrt{\frac{3}{\pi}},$$

which strengthens the estimate for $A_{n-1}(a)$. Again using

$$P_{n-2}(a) \le 2 \sum_{k=2}^{n} \sqrt{1 - a_k^2}\ A_{n-1}(a)\ ,$$

but now with a better estimate for $A_{n-1}(a)$, we may prove Theorem 1 in this case.

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The proof of Proposition 4 is technically involved, as already Ball's estimate $f(p) \le f(2) = 1$ for $p \ge 2$.

The complex analogue of Ball's function is

$$g(p) := \frac{p}{2}\frac{1}{2}\int_0^\infty |j_1(t)|^p \; t \; dt \; .$$

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Again

$$g(2) = 1, \; g(p) \leq g(2) \text{ for all } p \geq 2 \; .$$

But now

$$\lim_{p \to \infty} g(p) = g(2) = 1 \; ,$$

so no improvement of the constrained optimization technique would be possible by improving Ball's function estimate. However, it is not necessary in the complex case, as we have seen.

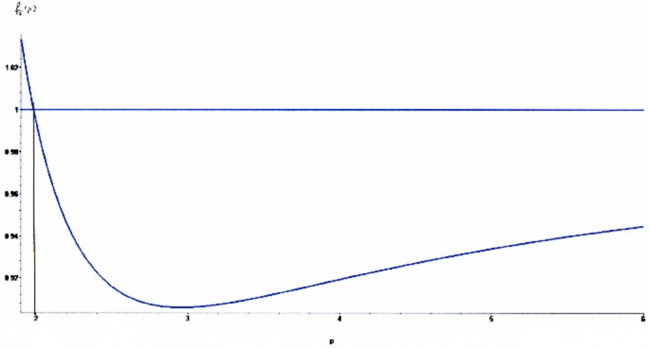

Abbildung: Complex analogue of Ball's function