Karlsruhe Institute of Technology

A reverse Minkowski-type inequality

Daniel Hug | Conference on Convex, Discrete and Integral Geometry, September 2019

JOINT WORK WITH KÁROLY BÖRÖCZKY

Conference on

Convex, Integral and Stochastic Geometry in Bad Herrenalb (near Karlsruhe) 6–11 September 2020

in honour of

Rolf Schneider (*1940) & Wolfgang Weil (1945-2018)

The **isoperimetric inequality** for a convex body $K \in \mathcal{K}_o^2$ states that

$$A(K) \leq \frac{1}{4\pi} L(K)^2$$

with equality iff K is a Euclidean 2-ball.

Since for $K, M \in \mathcal{K}_o^2$,

$$A(K + M) = A(K) + 2\, A(K, M) + A(M)$$

and

$$L(K + M)^2 = L(K)^2 + 2\, L(K)L(M) + L(M)^2,$$

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$$A(K, M) \leq c\, L(K)\, L(M) \qquad \text{with} \qquad c = \frac{1}{4\pi}\,?$$

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Choose $K = [o, e_1]$ and $M = [o, e_2]$. Then

$$A(K, M) = \frac{1}{2}(A(K + M) - A(K) - A(M)) = \frac{1}{2}$$

and

$$L(K) = L(M) = 2.$$

Hence, necessarily

$$c \geq \frac{A(K, M)}{L(K)\, L(M)} \geq \frac{1}{8} \geq \frac{1}{4\pi}.$$

This is in fact best possible.

Theorem (Betke, Weil '91)

If $K, M \in \mathcal{K}^2$, then

$$A(K, M) \leq \frac{1}{8} L(K)\, L(M)$$

with equality iff K and M are (possibly degenerate) orthogonal segments.

Proof. Use Choquet representation of convex sets (integral version of the Krein-Milman theorem). Characterization of the equality case is more subtle.

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Modified proof of the inequality. If $K = P$ is a polygon, then $P = \triangle_1 + \cdots + \triangle_m$ with (possibly degenerate) triangles $\triangle_i$.

- By approximation it suffices to show

$$A(P, M) \leq \frac{1}{8} L(P)\, L(M)$$

- By linearity, it is sufficient to consider $P = \triangle$.
- By homog. and transl. inv., we can assume that $B^2 = B_c(\triangle)$.

$$\Rightarrow A(\triangle, M) = \frac{1}{2} \int_{\mathbb{S}^1} h(\triangle, u)\, S(M, du) \leq \frac{1}{2} \cdot 1 \cdot L(M).$$

Lemma

If $\triangle$ is a triangle with circumball B^2, then $L(\triangle) \geq 4$.

$$\Rightarrow A(\triangle, M) \leq \frac{1}{2} \cdot \frac{1}{4} L(\triangle) \cdot L(M) = \frac{1}{8} L(\triangle)\, L(M).$$

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General dimensions: For $K, M \in \mathcal{K}^n$, Minkowski's inequality states that

$$V(K, M, \ldots, M)^n \geq V(K)\, V(M)^{n-1}.$$

Equality holds iff K, M are homothetic or lie in parallel hyperplanes or $\dim(M) \leq n - 2$.

In the plane, the inequality states that

$$V(K, M) \geq V(K)^{\frac{1}{2}}\, V(M)^{\frac{1}{2}}.$$

Apparently, this inequality **cannot** be reversed by inserting some constant.

However, the result by Betke & Weil '91 can be considered a reverse form of the planar Minkowski inequality.

Equivalently, their **planar** result can be stated in the form

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Reverse Minkowski-type inequality

Theorem (Böröczky & H. '19)

If $K, M \in \mathcal{K}^n$, then

$$V(K, M[n-1]) \leq \frac{1}{n} V_1(K) V_{n-1}(M).$$

If $\dim(K) \geq 1$ and $\dim(M) \geq n-1$, then equality holds iff K is a segment and M is contained in a hyperplane orthogonal to K.

In the following, we write $\mathcal{K}_i^n$ for the space of convex bodies which are at least i-dimensional.

Related work I

- Betke and Weil (1991) also proved that if $K \in \mathcal{K}^2$, then

$$V(K, -K) \leq \frac{\sqrt{3}}{18} L^2(K), \tag{1}$$

  and under the additional assumption that K is a polygon they showed that equality holds in (1) iff K is a regular triangle.

- Betke and Weil (1991) suggested to characterize the equality cases of (1) among all planar compact convex sets $K \in \mathcal{K}^2$. This goal will be achieved in a forthcoming manuscript.

- They also suggested to study sharp inequalities of the form

$$V(K_1[r_1], \ldots, K_\ell[r_\ell]) \leq c(r_1, \ldots, r_\ell)\, V_{r_1}(K_1) \cdots V_{r_\ell}(K_\ell).$$

  Inequ's of this type are known for **zonoids** (Schneider & H '11, w.i.p.).

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Related work II

- Saroglou, Soprunov, Zvavitch '18: If $K \in \mathcal{K}^n$ is such that

$$V(L_1, \dots, L_n)V(K) \le V(L_1, K[n-1])V(L_2, \dots, L_n, K)$$

for all $L_1, \dots, L_n \in \mathcal{K}^n$, then $K = \triangle$.

- They also prove the **Bezout-type** inequality

$$V(L_1, \dots, L_n)V(K) \le n \, V(L_1, K[n-1])V(L_2, \dots, L_n, K) \qquad \text{(BI)}$$

for all $K, L_1, \dots, L_n \in \mathcal{K}^n$. More general inequalities: Jian Xiao '19.

- (BI) is sharp, **but** for $K = B^n$, $L_1 =: K$, $L_2 = \dots = L_n =: M$ states

$$V(K, M[n-1]) \le \frac{2\kappa_{n-1}}{\kappa_n} \cdot \frac{1}{n} V_1(K) \, V_{n-1}(M)$$

with

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Theorem (Linhart '77, Böröczky & H. '19)

If $K \in \mathcal{K}_1^n$, then

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- This also prepares for stronger stability results.

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Recall

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If $\mathrm{dim}(K) \geq 1$ and $\mathrm{dim}(M) \geq n-1$, then equality holds iff K is a segment and M is contained in a hyperplane orthogonal to K.

Proof. We can assume that $c(K) = o$. Let $R(K)$ be the circumradius of K and $F(M)$ the surface area of M. Then

$$\begin{aligned} V(K, M[n-1]) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u)\, S_{n-1}(M, du) \\ &\leq \frac{1}{n} \, R(K) \, F(M) \qquad (1) \\ &\leq \frac{1}{n} \, \frac{1}{2} V_1(K) \, 2 V_{n-1}(M) \qquad (2) \\ &= \frac{1}{n} V_1(K) V_{n-1}(M). \end{aligned}$$

If equality holds, then equality holds in (2), since $V_{n-1}(M) > 0$. Hence $K = [-Re, Re]$ with $R = R(K)$ and for some $e \in \mathbb{S}^{n-1}$.
Then also (1) holds with equality, which yields

$$\int_{\mathbb{S}^{n-1}} |\langle u, e\rangle| \, S_{n-1}(M, du) = F(M).$$

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Recall

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If $K \in \mathcal{K}_1^n$, then

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with equality iff K is a segment.

In the proof, the following lemma is crucial.

For $z \in \mathbb{S}^{n-1}$ and $\alpha \in (0, \pi)$, let $B(z, \alpha) = \{x \in \mathbb{S}^{n-1} : \langle x, z \rangle \geq \cos \alpha\}$ be the **spherical cap** centered at z and of radius α.

Lemma (Monotonicity)

If $\alpha \in (0, \frac{\pi}{2}]$, $n \geq 2$, $z \in \mathbb{S}^{n-1}$ and $\Pi \subset B(z, \alpha)$ is compact and spherically starshaped with respect to z, then

$$\int_{\Pi} \langle z, u \rangle \, \mathcal{H}^{n-1}(du) \geq \frac{\int_{B(z,\alpha)} \langle z, u \rangle \, \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z, \alpha))} \cdot \mathcal{H}^{n-1}(\Pi).$$

If $z \in \mathrm{int}_s(\Pi)$, then equality holds iff $\Pi = B(z, \alpha)$.

Proof of the Theorem.

We can assume that B^n is the circumball of K, hence $R(K) = 1$.
Then the origin o is contained in the convex hull of $x_1, \ldots, x_k \in \mathbb{S}^{n-1} \cap K$ with $2 \leq k \leq n+1$.

For $i = 1, \ldots, k$, we define the Dirichlet-Voronoi cells

$$D_i = \{x \in \mathbb{S}^{n-1} : \langle x, x_i \rangle \geq \langle x, x_j \rangle \text{ for } j = 1, \ldots, k\} \subset B\left(x_i, \frac{\pi}{2}\right).$$

Then D_i is starshaped with respect to $x_i \in \text{int}_s(D_i)$ and

$$\sum_{i=1}^{k} \mathcal{H}^{n-1}(D_i) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}).$$

For $x \in D_i$, we have $h([x_1, \ldots, x_k], x) = \langle x, x_i \rangle$.

Proof of the Theorem.

We can assume that B^n is the circumball of K, hence $R(K) = 1$.
Then the origin o is contained in the convex hull of $x_1, \ldots, x_k \in \mathbb{S}^{n-1} \cap K$ with $2 \leq k \leq n+1$.

For $i = 1, \ldots, k$, we define the Dirichlet-Voronoi cells

$$D_i = \{x \in \mathbb{S}^{n-1} : \langle x, x_i \rangle \geq \langle x, x_j \rangle \text{ for } j = 1, \ldots, k\} \subset B\left(x_i, \frac{\pi}{2}\right).$$

Then D_i is starshaped with respect to $x_i \in \text{int}_s(D_i)$ and

$$\sum_{i=1}^{k} \mathcal{H}^{n-1}(D_i) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}).$$

For $x \in D_i$, we have $h([x_1, \ldots, x_k], x) = \langle x, x_i \rangle$.

Then,

$$V_1(K) \geq V_1([x_1, \ldots, x_k]) = \frac{n}{\kappa_{n-1}} V([x_1, \ldots, x_k], B^n[n-1]) \tag{2}$$

$$= \frac{1}{\kappa_{n-1}} \sum_{i=1}^{k} \int_{D_i} \langle x, x_i \rangle \, \mathcal{H}^{n-1}(dx)$$

$$\geq \frac{1}{\kappa_{n-1}} \sum_{i=1}^{k} \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(D_i) = 2. \tag{3}$$

If $V_1(K) = 2$, then equality in (2) yields that $K = [x_1, \ldots, x_k]$.

Moreover, by the Lemma strict inequality holds in (3) if $D_i \neq B\left(x_i, \frac{\pi}{2}\right)$ for some $i \in \{1, \ldots, k\}$.

Hence, if equality holds, we must have $k = 2$ and $K = [x_1, x_2]$.

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Stability I

$$V_1(K) \geq 2R(K).$$

Theorem (Böröczky & H. '19)

Let $K \in \mathcal{K}_1^n$. For some small $\varepsilon > 0$, suppose that

$$V_1(K) \leq (2+\varepsilon)R(K).$$

Then there exists a segment s of length $(2-\gamma_1\varepsilon)R(K)$ such that

$$s \subset K \subset s + \gamma_2 R(K)\sqrt{\varepsilon}B^n,$$

where $\gamma_1, \gamma_2 > 0$ are constants depending on n.

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Stability II

$$V(K, M[n-1]) \leq \frac{1}{n} V_1(K) \, V_{n-1}(M).$$

Theorem (Böröczky & H. '19)

Let $K \in \mathcal{K}_1^n$ and $M \in \mathcal{K}_{n-1}^n$. For some $\varepsilon \in (0, \varepsilon_0)$, suppose that

$$V(K, M[n-1]) \geq (1-\varepsilon) \frac{1}{n} V_1(K) V_{n-1}(M).$$

Then there exists $e \in \mathbb{S}^{n-1}$, a segment s of length $(2 - \gamma_1 \varepsilon) R(K)$ parallel to e such that $s \subset K \subset s + \gamma_2 R(K) \sqrt{\varepsilon} B^n$, and there is a $f \in \mathbb{S}^{n-1}$ s.t.

$$h_M(f) + h_M(-f) \leq \gamma_2 \, r \, \varepsilon^{\frac{1}{4}}, \qquad \langle e, f \rangle \geq 1 - \gamma_2 \sqrt{\varepsilon},$$

where r is the maximal radius of an $(n-1)$-ball in $M|e^{\perp}$, and $\gamma_1, \gamma_2, \varepsilon_0 > 0$ are constants depending on n.

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Stability III

$$\int_{\mathbb{S}^{n-1}} |\langle e, u\rangle| \, S_{n-1}(M, du) \leq 2V_{n-1}(M), \qquad e \in \mathbb{S}^{n-1}.$$

Proposition (Böröczky & H. '19)

Let $\varepsilon \in (0, \frac{1}{2}\left(\frac{1}{2n}\right)^n)$ and $e \in \mathbb{S}^{n-1}$. Suppose that $M \in \mathcal{K}^n_{n-1}$ is such that

$$\int_{\mathbb{S}^{n-1}} |\langle e, u\rangle| \, S_{n-1}(M, du) \geq (1-\varepsilon) 2V_{n-1}(M).$$

Then there is some $f \in \mathbb{S}^{n-1}$ such that

$$h_M(f) + h_M(-f) \leq c_7 r\sqrt{\varepsilon}, \qquad \langle e, f\rangle \geq 1 - c_8\varepsilon,$$

where $c_7 \leq 48n^2\sqrt{6}^n$, $c_8 \leq (10n)^4(2n)^n$ and r is as before.

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An application to random tessellations

Geometric stability results have been useful also in stochastic geometry.

Theorem (Böröczky & H. ’19)

Let Z_0 denote the zero cell of a stationary and isotropic Poisson hyperplane tessellation in $\mathbb{R}^n$ with intensity $\lambda > 0$. Then there is a constant c_0 (depending on n) such that the following holds. If $\varepsilon > 0$ and $0 < a < b \leq \infty$, then

$$\mathbb{P}(\vartheta(Z_0) \geq \varepsilon \mid R(Z_0) \in [a, b)) \leq c \exp\{c_0 \, \varepsilon \, a \, \lambda\},$$

where c is a constant which depends on n, ε.

This completes the discussion of an application of a general result in Schneider & H ’07 to a specific functional.

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in honour of

Rolf Schneider (*1940) & Wolfgang Weil (1945-2018)