

The Total Curvature and Betti Numbers of Complex Projective Manifolds

Convex, Discrete and Integral Geometry
Friedrich-Schiller-Universität Jena

Joseph Ansel Hoisington
University of Georgia

19th September, 2019

Chern and Lashof defined the total absolute curvature as an invariant of a manifold immersed in Euclidean space:

Chern and Lashof defined the total absolute curvature as an invariant of a manifold immersed in Euclidean space:

The integral, over the unit normal vectors to the immersion, of the absolute value of det. of the second fundamental form:

$$T(M^n) := \frac{1}{\text{Vol}(S^{N-1})} \int_{\nu^1 M} |\det(A_{\vec{u}})| d\text{Vol}_{\nu^1 M}$$

$\nu^1 M$ is the unit normal bundle of the immersion, $A_{\vec{u}}$ is the second fundamental form in the normal direction \vec{u} .

For any closed manifold M isometrically immersed in Euclidean space \mathbb{R}^N , they were able to prove:

The Chern-Lashof Theorems

Let M^n be a closed manifold immersed in \mathbb{R}^N :

The Chern-Lashof Theorems

Let M^n be a closed manifold immersed in \mathbb{R}^N :

Theorem (First Chern-Lashof Theorem)

Let β_i be the i^{th} Betti number of M ; coefficients can be the integers or any field. Then:

$$\sum_{i=0}^n \beta_i \leq T(M^n)$$

In particular, $T(M^n) \geq 2$.

The Chern-Lashof Theorems

Let M^n be a closed manifold immersed in \mathbb{R}^N :

Theorem (First Chern-Lashof Theorem)

Let β_i be the i^{th} Betti number of M ; coefficients can be the integers or any field. Then:

$$\sum_{i=0}^n \beta_i \leq T(M^n)$$

In particular, $T(M^n) \geq 2$.

Theorem (Second Chern-Lashof Theorem)

If $T(M^n) < 3$, then M is homeomorphic to a sphere.

The Chern-Lashof Theorems

Let M^n be a closed manifold immersed in \mathbb{R}^N :

Theorem (First Chern-Lashof Theorem)

Let β_i be the i^{th} Betti number of M ; coefficients can be the integers or any field. Then:

$$\sum_{i=0}^n \beta_i \leq T(M^n)$$

In particular, $T(M^n) \geq 2$.

Theorem (Second Chern-Lashof Theorem)

If $T(M^n) < 3$, then M is homeomorphic to a sphere.

Theorem (Third Chern-Lashof Theorem)

$T(M^n) = 2$ precisely when M is embedded as the boundary of a convex subset in an affine subspace \mathbb{R}^{n+1} of \mathbb{R}^N .

Chern, Shiing-shen, and Lashof, Richard K.:

[CL57] "On the total curvature of immersed manifolds" *American Journal of Mathematics* 79.2 (1957): 306-318,

[CL58] "On the total curvature of immersed manifolds II" *Michigan Mathematical Journal* 5 (1958): 5-12.

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$, of complex dimension m :

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$, of complex dimension m :

Theorem (H.)

Let $\beta_i(M)$ be its Betti numbers with real coefficients. Then:

$$\sum_{i=0}^{2m} \beta_i(M) \leq \left(\frac{m+1}{2}\right) T_{\mathbb{C}P^N}(M).$$

In particular, $T_{\mathbb{C}P^N}(M) \geq 2$.

The inequality above follows from several other inequalities between the total absolute curvature and Betti numbers of complex projective manifolds, which are generally stronger - time permitting, these will be explained below.

Background for the Chern-Lashof Theorems: Fenchel and Fáry-Milnor

In their first paper on the subject, Chern and Lashof cite the theorems of Fenchel and Fáry-Milnor as a source of motivation:

In their first paper on the subject, Chern and Lashof cite the theorems of Fenchel and Fáry-Milnor as a source of motivation:

Theorem (Fenchel's Theorem, [Fe29])

Let γ be a smooth closed curve in the Euclidean space \mathbb{R}^3 . Let κ be the curvature of γ .

Then $\int_{\gamma} \kappa \geq 2\pi$, with equality precisely for plane convex curves.

In their first paper on the subject, Chern and Lashof cite the theorems of Fenchel and Fáry-Milnor as a source of motivation:

Theorem (Fenchel's Theorem, [Fe29])

Let γ be a smooth closed curve in the Euclidean space \mathbb{R}^3 . Let κ be the curvature of γ .

Then $\int_{\gamma} \kappa \geq 2\pi$, with equality precisely for plane convex curves.

Theorem (Fáry-Milnor Theorem, [Fá49], [Mi50])

Let γ be a smooth closed curve embedded in \mathbb{R}^3 , and suppose that $\int_{\gamma} \kappa \leq 4\pi$.

Then γ is unknotted.

[Fe29] W. Fenchel "Über Krümmung und Windung Geschlossener Raumkurven", *Mathematische Annalen* 101.1 (1929): 238-252

[Fá49] I. Fáry "Sur la Courbure Totale d'une Courbe Gauche Faisant un Noeud", *Bull. Soc. Math. France* 77 (1949): 128-138

[Mi50] J. Milnor "On the Total Curvature of Knots", *Annals of Mathematics* (1950): 248-257

Along these lines, we will also prove:

Along these lines, we will also prove:

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$.

If $T_{\mathbb{C}P^N}(M) < 4$, then in fact $T_{\mathbb{C}P^N}(M) = 2$. This occurs precisely if M is a linearly embedded complex projective subspace.

The upper bound 4 and the strict inequality in this result are the best possible - to be explained below.

Along these lines, we will also prove:

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$.

If $T_{\mathbb{C}P^N}(M) < 4$, then in fact $T_{\mathbb{C}P^N}(M) = 2$. This occurs precisely if M is a linearly embedded complex projective subspace.

The upper bound 4 and the strict inequality in this result are the best possible - to be explained below.

Corollary (of the proof)

Linear subspaces are the only complex projective manifolds M whose spherical pre-images via the Hopf fibration are homology spheres.

We define the total absolute curvature of a complex projective manifold, of complex dimension m , as follows:

$$\frac{2}{\text{Vol}(\mathbb{C}P^N)} \int_{v^1 M} \int_0^{\frac{\pi}{2}} \left| \det(\cos(r)Id_{T_p M} - \sin(r)A_{\vec{u}}) \right| \cos(r) \sin^{(2N-2m-1)}(r) dr d\text{Vol}_{v^1 M}(\vec{u})$$

$v^1 M$ is the unit normal bundle of M . $A_{\vec{u}}$ is the second fundamental form of the normal vector \vec{u} , and $Id_{T_p M}$ is the identity transformation of the tangent space to M at its base point p .

Two interpretations of the total curvature of a complex projective manifold:

$$1.) T_{\mathbb{C}P^N}(M) = \frac{2}{\text{Vol}(\mathbb{C}P^N)} \int_{v < \frac{\pi}{2} M} |\det(d\text{Exp}^\perp)| d\text{Vol}_{v < \frac{\pi}{2} M}$$

$$2.) T_{\mathbb{C}P^N}(M) = \frac{2}{\text{Vol}(\mathbb{C}P^N)} \int_{\mathbb{C}P^N} \#(\text{Exp}^\perp)^{-1}(q) d\text{Vol}_{\mathbb{C}P^N}$$

These are equivalent to the interpretation of Chern and Lashof's invariant for submanifolds of Euclidean space.

The Chern-Lashof theorems are related to the Gauss-Bonnet-Chern theorem - in particular, to the following fact about a closed manifold M immersed in Euclidean space:

$$\chi(M) = \sum_{i=0}^n (-1)^i \beta_i = \frac{1}{\text{Vol}(S^{N-1})} \int_{v^1 M} \det(A_{\vec{u}}) d\text{Vol}_{v^1 M}.$$

(The first Chern-Lashof theorem says:)

$$\sum_{i=0}^n \beta_i \leq \frac{1}{\text{Vol}(S^{N-1})} \int_{v^1 M} |\det(A_{\vec{u}})| d\text{Vol}_{v^1 M}.$$

Total Curvature and the Gauss-Bonnet-Chern Theorem in Complex Projective Space

In the complex projective setting, we have:

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in complex projective space. Let TM be the tangent bundle of M and \tilde{L} the line bundle on M pulled back from $\mathcal{O}(-1) \in \text{Pic}(\mathbb{C}P^N)$ by the immersion. Then:

$$\frac{1}{\text{Vol}(\mathbb{C}P^N)} \int_{v < \frac{\pi}{2} M} \det(d\text{Exp}^\perp) d\text{Vol}_{v < \frac{\pi}{2} M} = e(TM \otimes \tilde{L}),$$

where $e(TM \otimes \tilde{L})$ is the Euler number of $TM \otimes \tilde{L}$.

(The theorem above says:)

$$\sum_{i=0}^{2m} \beta_i(M) \leq \frac{m+1}{\text{Vol}(\mathbb{C}P^N)} \int_{v < \frac{\pi}{2} M} |\det(d\text{Exp}^\perp)| d\text{Vol}_{v < \frac{\pi}{2} M}.$$

Intrinsic Total Curvature for Complex Projective Manifolds

Calabi proved in [Ca53] that if a Kähler metric is induced by a holomorphic immersion into $\mathbb{C}P^N$, even locally, then that immersion is unique, up to the holomorphic isometries of $\mathbb{C}P^N$.

So although it is calculated with the second fundamental form of M in $\mathbb{C}P^N$, the total curvature is actually part of its intrinsic geometry.

Calabi proved in [Ca53] that if a Kähler metric is induced by a holomorphic immersion into $\mathbb{C}P^N$, even locally, then that immersion is unique, up to the holomorphic isometries of $\mathbb{C}P^N$.

So although it is calculated with the second fundamental form of M in $\mathbb{C}P^N$, the total curvature is actually part of its intrinsic geometry.

Theorem (H.)

Let Σ be a curve in $\mathbb{C}P^2$, with K the sectional curvature of its projectively induced metric.

$$\text{Then } T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} \frac{(K-4)^2 + 4}{(6-K)} dA_{\Sigma}.$$

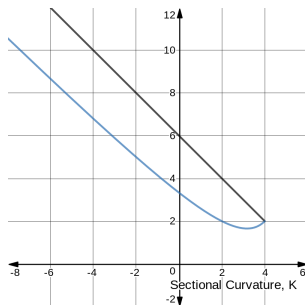
Note that $K \leq 4$, because the holomorphic sectional curvature of a Kähler submanifold is bounded above by the holomorphic sectional curvature of the ambient metric.

[Ca53] E. Calabi "Isometric Imbedding of Complex Manifolds." *Annals of Mathematics* (1953): 1-23.

Theorem (H.)

Let Σ be a curve in $\mathbb{C}P^2$, K the curvature of its projectively induced metric:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} \frac{(K-4)^2 + 4}{(6-K)} dA_{\Sigma}$$



$$f(K) = \frac{(K-4)^2 + 4}{(6-K)}, f(K) = 6 - K$$

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$.

If $T_{\mathbb{C}P^N}(M) < 4$, then in fact $T_{\mathbb{C}P^N}(M) = 2$. This occurs precisely if M is a linearly embedded complex projective subspace.

The strict inequality and the constant 4 in this theorem are the best possible:

Applications of the Intrinsic Formula for the Total Absolute Curvature of a Plane Curve

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$.

If $T_{\mathbb{C}P^N}(M) < 4$, then in fact $T_{\mathbb{C}P^N}(M) = 2$. This occurs precisely if M is a linearly embedded complex projective subspace.

The strict inequality and the constant 4 in this theorem are the best possible:

The curve \mathcal{F} described by $z_0^2 + z_1^2 + z_2^2 = 0$ in $\mathbb{C}P^2$ is isometric to a round sphere with constant curvature 2.

We can calculate its total absolute curvature as follows:

Theorem (H.)

Let M be a closed complex manifold holomorphically immersed in the projective space $\mathbb{C}P^N$.

If $T_{\mathbb{C}P^N}(M) < 4$, then in fact $T_{\mathbb{C}P^N}(M) = 2$. This occurs precisely if M is a linearly embedded complex projective subspace.

The strict inequality and the constant 4 in this theorem are the best possible:

The curve \mathcal{F} described by $z_0^2 + z_1^2 + z_2^2 = 0$ in $\mathbb{C}P^2$ is isometric to a round sphere with constant curvature 2.

We can calculate its total absolute curvature as follows:

$$T(\mathcal{F}) = \frac{1}{\pi} \int_{\Sigma} \frac{(2-4)^2 + 4}{(6-2)} dA_{\Sigma} = \frac{1}{\pi} \times 2\pi \times 2 = 4.$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma}$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \leq \frac{1}{\pi} \int_{\Sigma} (6 - K) dA_{\Sigma}$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \leq \frac{1}{\pi} \int_{\Sigma} (6 - K) dA_{\Sigma} = \frac{1}{\pi} \int_{\Sigma} 6 dA_{\Sigma} - \frac{1}{\pi} \int_{\Sigma} K dA_{\Sigma}$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \leq \frac{1}{\pi} \int_{\Sigma} (6 - K) dA_{\Sigma} = \frac{1}{\pi} \int_{\Sigma} 6 dA_{\Sigma} - \frac{1}{\pi} \int_{\Sigma} K dA_{\Sigma}$$

$= 6d - 4 + 2(d-1)(d-2)$ (using the Gauss-Bonnet formula for the curvature integral,)

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Proof:

Fact: a (smooth) curve of degree d in $\mathbb{C}P^2$ is a compact Riemann surface of genus $\frac{(d-1)(d-2)}{2}$ and has area $d\pi$ ($= d \times \text{Area}(\mathbb{C}P^1)$).

Let $f(K) = \frac{(K-4)^2+4}{(6-K)}$, the function of sectional curvature which we integrate over Σ to calculate $T(\Sigma)$. Then $f(K) \leq 6 - K$ for $K \leq 4$, so:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \leq \frac{1}{\pi} \int_{\Sigma} (6 - K) dA_{\Sigma} = \frac{1}{\pi} \int_{\Sigma} 6 dA_{\Sigma} - \frac{1}{\pi} \int_{\Sigma} K dA_{\Sigma}$$

$$\begin{aligned} &= 6d - 4 + 2(d-1)(d-2) \text{ (using the Gauss-Bonnet formula for the curvature integral,)} \\ &= 2d^2. \end{aligned}$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Also: $f(K) = \frac{(K-4)^2+4}{(6-K)}$ is convex for $K \leq 4$.

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Also: $f(K) = \frac{(K-4)^2+4}{(6-K)}$ is convex for $K \leq 4$. Jensen's inequality then implies that:

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Also: $f(K) = \frac{(K-4)^2+4}{(6-K)}$ is convex for $K \leq 4$. Jensen's inequality then implies that:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma}$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Also: $f(K) = \frac{(K-4)^2+4}{(6-K)}$ is convex for $K \leq 4$. Jensen's inequality then implies that:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \geq df \left(\frac{1}{d\pi} \int_{\Sigma} K dA_{\Sigma} \right).$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Also: $f(K) = \frac{(K-4)^2+4}{(6-K)}$ is convex for $K \leq 4$. Jensen's inequality then implies that:

$$T(\Sigma) = \frac{1}{\pi} \int_{\Sigma} f(K) dA_{\Sigma} \geq df \left(\frac{1}{d\pi} \int_{\Sigma} K dA_{\Sigma} \right).$$

Using the Gauss-Bonnet formula for the curvature integral again, we have:

$$T(\Sigma) \geq 2d^2 - 4d + 4.$$

(The inequality between total curvature and Betti numbers implies:)

$$T(\Sigma) \geq 2d^2 - 6d + 6.$$

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

Equality in the first inequality requires equality in Jensen's inequality. This is possible only if K is constant - this is possible only for curves homeomorphic to S^2 (see [Hu00]), hence for $d = 1$ or 2 .

In case $d = 2$, this occurs only for curves congruent to \mathcal{F} - other conics will have total absolute curvature strictly greater than 4.

Equality in the second inequality requires $f(K) = 6 - K$. This is possible only if $K = 4$, which occurs only when $d = 1$.

[Hu00] D. Hulin "Kähler-Einstein Metrics and Projective Embeddings" *The Journal of Geometric Analysis* 10.3 (2000): 525-528

Theorem (H.)

Let Σ be a smooth curve in $\mathbb{C}P^2$. Then the degree of Σ is the unique natural number d such that $2d^2 - 4d + 4 \leq T(\Sigma) \leq 2d^2$.

The lower bound for degree $d + 1$ curves, $2(d + 1)^2 - 4(d + 1) + 4$, is greater than the upper bound for degree d curves, $2d^2$, by 2.








This implies that the total curvature determines the degree.

In contrast, the Gauss-Bonnet integral is the same for (smooth) plane curves of degrees 1 and 2.

$T(\Sigma)$ can also distinguish some geometrically distinct curves of the same degree.

So, in this sense, total absolute curvature is a slightly stronger invariant for plane curves than total (non-absolute) curvature.

Thank You!

-  Calabi, Eugenio. *Isometric imbedding of complex manifolds*, Annals of Mathematics (1953): 1-23.
-  Chern, Shiing-shen, and Richard K. Lashof. *On the total curvature of immersed manifolds* American Journal of Mathematics 79.2 (1957): 306-318.
-  Chern, Shiing-shen, and Richard K. Lashof. *On the total curvature of immersed manifolds II* Michigan Math. Journal 5 (1958): 5-12.
-  I. Fáry *Sur la Courbure Totale d'une Courbe Gauche Faisant un Noeud*, Bull. Soc. Math. France 77 (1949): 128-138
-  W. Fenchel *Über Krümmung und Windung Geschlossener Raumkurven*, Mathematische Annalen 101.1 (1929): 238-252
-  D. Hulin *Kähler-Einstein Metrics and Projective Embeddings* The Journal of Geometric Analysis 10.3 (2000): 525-528
-  J. Milnor *On the Total Curvature of Knots*, Annals of Mathematics (1950): 248-257

These results can be found in:

J. A. Hoisington "On the Total Curvature and Betti Numbers of Complex Projective Manifolds", arXiv preprint : 1807.11625 (2018)
Accepted - Geometry & Topology.