Randomized Urysohn–type inequalities

Thomas Hack[1] jointly with Peter Pivovarov[2]

[1]Vienna University of Technology

[2]University of Missouri

Convex, Discrete and Integral Geometry

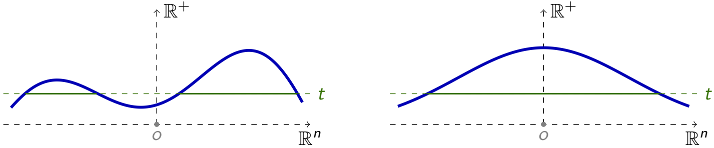
Friedrich–Schiller–Universität,

Jena, September 2019

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Rearrangements

$f \in L^1(\mathbb{R}^n)$, $f^\star$ *symmetric decreasing rearrangement*

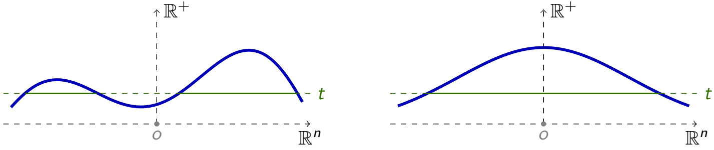


$\{f^\star > t\}$ is an open ball around o such that

$$\mathrm{vol}_n(\{f^\star > t\}) = \mathrm{vol}_n(\{f > t\})$$

Rearrangements

$f \in L^1(\mathbb{R}^n)$, $f^\star$ *symmetric decreasing rearrangement*



$\{f^\star > t\}$ is an open ball around o such that

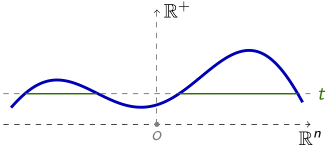
$$\mathrm{vol}_n(\{f^\star > t\}) = \mathrm{vol}_n(\{f > t\})$$

For $\Psi\colon (\mathbb{R}^n)^N \to \mathbb{R}^+$, $f_1, \ldots, f_N \in L^1(\mathbb{R}^n)$, set

$$I_\Psi(f_1, \ldots, f_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Psi(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i)\, dx_1 \ldots dx_N.$$

Rearrangements

$f \in L^1(\mathbb{R}^n)$, $f^\star$ *symmetric decreasing rearrangement*



$\{f^\star > t\}$ is an open ball around o such that

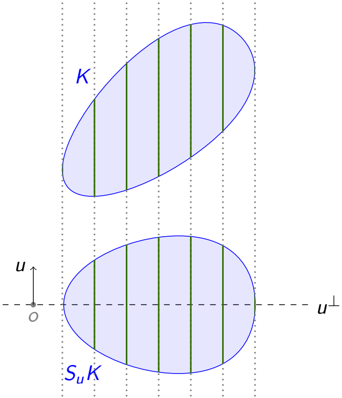
$$\mathrm{vol}_n(\{f^\star > t\}) = \mathrm{vol}_n(\{f > t\})$$

For $\Psi\colon (\mathbb{R}^n)^N \to \mathbb{R}^+$, $f_1, \dots, f_N \in L^1(\mathbb{R}^n)$, set

$$I_\Psi(f_1, \dots, f_N) := \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \Psi(x_1, \dots, x_N) \prod_{i=1}^{N} f_i(x_i)\, dx_1 \dots dx_N.$$

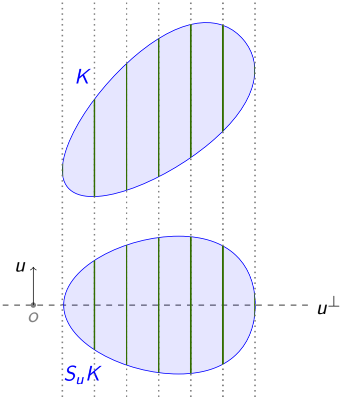
<u>Goal</u>: Find Ψ such that $I_\Psi(f_1, \dots, f_N) \ge I_\Psi(f_1^\star, \dots, f_N^\star)$ $[\le]$

Steiner symmetrization



$$(S_u f)(x) := \int_0^\infty \mathbb{1}_{S_u\{f>t\}}(x)\, dt$$

Steiner symmetrization



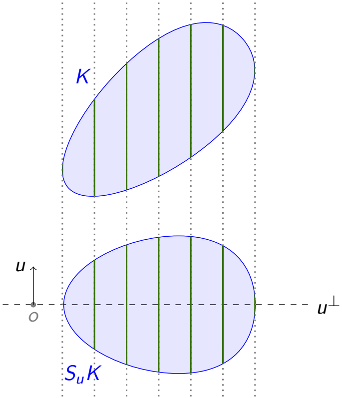
$$(S_u f)(x) := \int_0^\infty \mathbb{1}_{S_u\{f>t\}}(x)\, dt$$

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(S_u f_1, \ldots, S_u f_N)$$

for all $u \in \mathbb{S}^{n-1}$ implies

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(f_1^\star, \ldots, f_N^\star).$$

Steiner symmetrization



$$(S_u f)(x) := \int_0^\infty \mathbb{1}_{S_u\{f>t\}}(x)\, dt$$

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(S_u f_1, \ldots, S_u f_N)$$

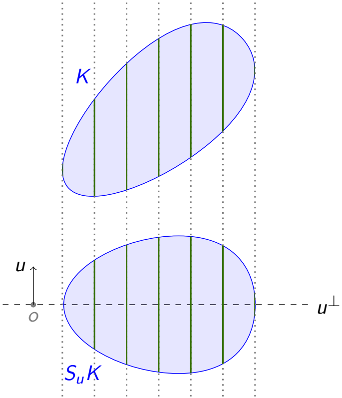
for all $u \in \mathbb{S}^{n-1}$ implies

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(f_1^\star, \ldots, f_N^\star).$$

- $\Psi(x_1, \ldots, x_N) = V_j(\mathrm{conv}\{x_1, \ldots, x_N\}), \quad j \in \{1, \ldots, n\}$

  Groemer '74, Pfiefer '82, Hartzoulaki, Paouris '03, Paouris, Pivovarov '17

Steiner symmetrization



$$(S_u f)(x) := \int_0^\infty \mathbb{1}_{S_u\{f>t\}}(x)\, dt$$

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(S_u f_1, \dots, S_u f_N)$$

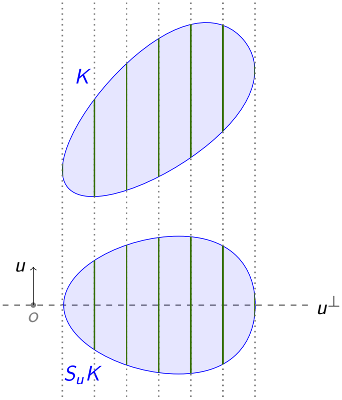
for all $u \in \mathbb{S}^{n-1}$ implies

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(f_1^\star, \dots, f_N^\star).$$

- $\Psi(x_1, \dots, x_N) = \mathrm{vol}_n(\mathrm{conv}\{\pm x_1, \dots, \pm x_N\}^\circ)$

  Cordero–Erausquin, Fradelizi, Paouris, Pivovarov '15

Steiner symmetrization



$$(S_u f)(x) := \int_0^\infty \mathbb{1}_{S_u\{f>t\}}(x)\,dt$$

$$I_\Psi(f_1,\dots,f_N) \geq I_\Psi(S_u f_1,\dots,S_u f_N)$$

for all $u \in \mathbb{S}^{n-1}$ implies

$$I_\Psi(f_1,\dots,f_N) \geq I_\Psi(f_1^\star,\dots,f_N^\star).$$

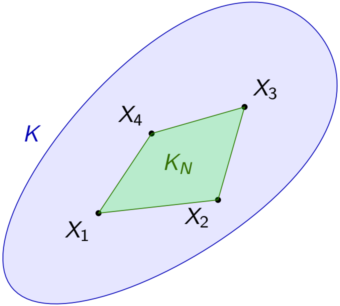
- $\Psi(x_1,\dots,x_N) = \mathrm{vol}_n\left(\bigcap_{i=1}^N B_r(x_i)\right), \quad r>0$

  Paouris, Pivovarov '17

Randomized isoperimetric inequalities

$X_1, \ldots, X_N \sim \mathrm{unif}(K)$, $Y_1, \ldots, Y_N \sim \mathrm{unif}(B_K)$, independent

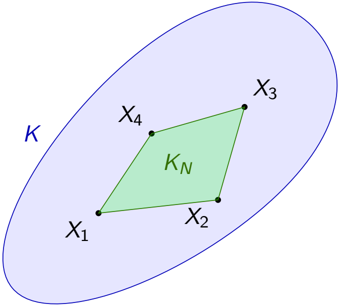
$$K_N := \mathrm{conv}\{X_1, \ldots, X_N\}, \quad (B_K)_N := \mathrm{conv}\{Y_1, \ldots, Y_N\}$$



Randomized isoperimetric inequalities

$X_1, \ldots, X_N \sim \mathrm{unif}(K)$, $Y_1, \ldots, Y_N \sim \mathrm{unif}(B_K)$, independent

$$K_N := \mathrm{conv}\{X_1, \ldots, X_N\}, \quad (B_K)_N := \mathrm{conv}\{Y_1, \ldots, Y_N\}$$



Empirical inequalities:

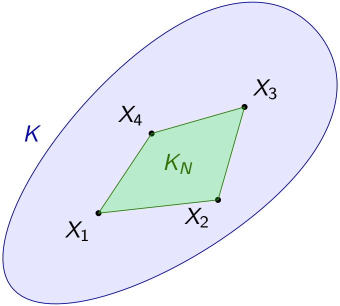
$$\mathbb{E}V_j(K_N) \geq \mathbb{E}V_j((B_K)_N),$$

$j \in \{1, \ldots, n\}$

Randomized isoperimetric inequalities

$X_1, \ldots, X_N \sim \mathrm{unif}(K)$, $Y_1, \ldots, Y_N \sim \mathrm{unif}(B_K)$, independent

$$K_N := \mathrm{conv}\{X_1, \ldots, X_N\}, \quad (B_K)_N := \mathrm{conv}\{Y_1, \ldots, Y_N\}$$



Empirical inequalities:

$$\mathbb{E}V_j(K_N) \geq \mathbb{E}V_j((B_K)_N),$$

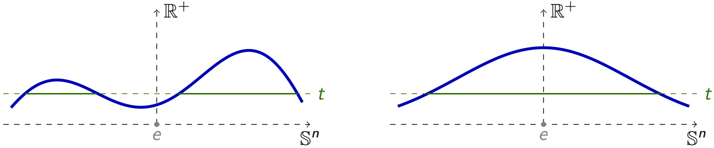
$j \in \{1, \ldots, n\}$

$K_N \to K$ a.s. as $N \to \infty$ empirical $\Rightarrow$ deterministic

Paouris, Pivovarov '17

Rearrangements

$f \in L^1(\mathbb{S}^n)$, $f^\star$ *symmetric decreasing rearrangement*



$\{f^\star > t\}$ is an open cap around e such that

$$\sigma_n(\{f^\star > t\}) = \sigma_n(\{f > t\})$$

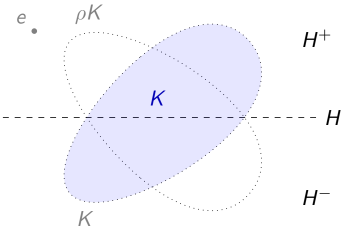
For $\Psi\colon (\mathbb{S}^n)^N \to \mathbb{R}^+$, $f_1, \dots, f_N \in L^1(\mathbb{S}^n)$, set

$$I_\Psi(f_1, \dots, f_N) := \int_{\mathbb{S}^n} \dots \int_{\mathbb{S}^n} \Psi(x_1, \dots, x_N) \prod_{i=1}^N f_i(x_i)\, dx_1 \dots dx_N.$$

Goal: Find Ψ such that $I_\Psi(f_1, \dots, f_N) \geq I_\Psi(f_1^\star, \dots, f_N^\star)$ $\quad [\leq]$

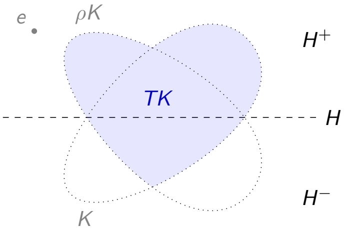
Two-point symmetrization

Hemisphere $H \in \mathcal{S}^n_{n-1}$, reflection $\rho\colon \mathbb{S}^n \to \mathbb{S}^n$



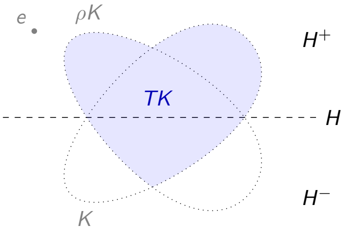
Two-point symmetrization

Hemisphere $H \in \mathcal{S}^n_{n-1}$, reflection $\rho\colon \mathbb{S}^n \to \mathbb{S}^n$



Two-point symmetrization

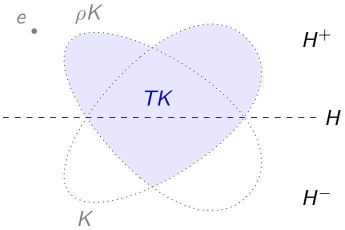
Hemisphere $H \in \mathcal{S}^n_{n-1}$, reflection $\rho\colon \mathbb{S}^n \to \mathbb{S}^n$



$$(Tf)(x) := \int_0^\infty \mathbb{1}_{T\{f>t\}}(x)\, dt$$

Two-point symmetrization

Hemisphere $H \in \mathcal{S}^n_{n-1}$, reflection $\rho\colon \mathbb{S}^n \to \mathbb{S}^n$



$$(Tf)(x) := \int_0^\infty \mathbb{1}_{T\{f>t\}}(x)\, dt$$

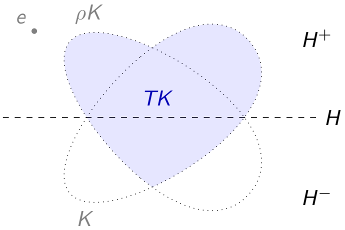
$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(Tf_1, \dots, Tf_N)$$

for all $H \in \mathcal{S}^n_{n-1}$ implies

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(f_1^\star, \dots, f_N^\star).$$

Two-point symmetrization

Hemisphere $H \in \mathcal{S}^n_{n-1}$, reflection $\rho \colon \mathbb{S}^n \to \mathbb{S}^n$



$$(Tf)(x) := \int_0^\infty \mathbb{1}_{T\{f>t\}}(x)\, dt$$

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(Tf_1, \dots, Tf_N)$$

for all $H \in \mathcal{S}^n_{n-1}$ implies

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(f_1^\star, \dots, f_N^\star).$$

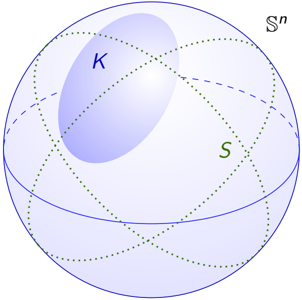
- $\Psi(x_1, \dots, x_N) = \prod_{i<j} \kappa_{ij}(d_{\mathbb{S}^n}(x_i, x_j))$, κ_{ij} decreasing

  Burchard, Schmuckenschläger '01, Morpurgo '02

Randomized Urysohn inequality

$K \in \mathcal{K}(\mathbb{S}^n) = \{K \subseteq \mathbb{S}^n \mid K \text{ compact, proper, convex, } \operatorname{int} K \neq \emptyset\}$:

$$U_1(K) := \int_{\mathcal{S}^n_{n-1}} \chi(K \cap S)\, dS$$



Randomized Urysohn inequality

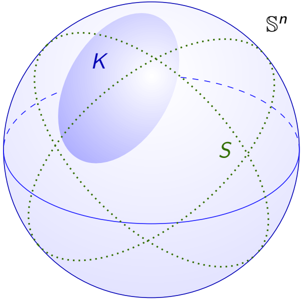
$K \in \mathcal{K}(\mathbb{S}^n) = \{K \subseteq \mathbb{S}^n \,|\, K \text{ compact, proper, convex, } \operatorname{int} K \neq \emptyset\}$:

$$U_1(K) := \int_{\mathcal{S}^n_{n-1}} \chi(K \cap S)\, dS$$

Theorem (H., Pivovarov)

$\Psi(x_1, \ldots, x_N) = U_1(\operatorname{conv}\{x_1, \ldots, x_N\})$

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(f_1^\star, \ldots, f_N^\star)$$



Randomized Urysohn inequality

$K \in \mathcal{K}(\mathbb{S}^n) = \{K \subseteq \mathbb{S}^n \,|\, K \text{ compact, proper, convex, } \operatorname{int} K \neq \emptyset\}$:

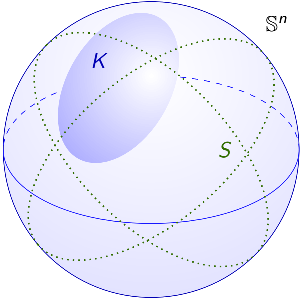
$$U_1(K) := \int_{\mathcal{S}^n_{n-1}} \chi(K \cap S)\, dS$$

Theorem (H., Pivovarov)

$\Psi(x_1, \ldots, x_N) = U_1(\operatorname{conv}\{x_1, \ldots, x_N\})$

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(f_1^\star, \ldots, f_N^\star)$$

- As $N \to \infty$, $U_1(K) \geq U_1(C_K)$
  Gao, Hug, Schneider '02
- Empirical Blaschke–Santaló via $1 - U_1(K) \simeq \sigma_n(K^*)$



Randomized Urysohn inequality

$K \in \mathcal{K}(\mathbb{S}^n) = \{K \subseteq \mathbb{S}^n \mid K \text{ compact, proper, convex, } \operatorname{int} K \neq \emptyset\}$:

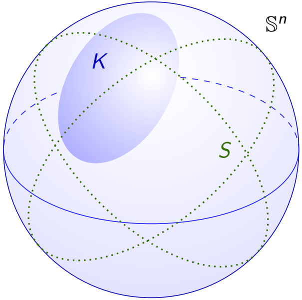
$$U_1(K) := \int_{\mathcal{S}^n_{n-1}} \chi(K \cap S)\, dS$$

Theorem (H., Pivovarov)

$\Psi(x_1, \dots, x_N) = U_1(\operatorname{conv}\{x_1, \dots, x_N\})$

$$I_\Psi(f_1, \dots, f_N) \geq I_\Psi(f_1^\star, \dots, f_N^\star)$$

- As $N \to \infty$, $U_1(K) \geq U_1(C_K)$
  Gao, Hug, Schneider '02
- Empirical Blaschke–Santaló via
  $1 - U_1(K) \simeq \sigma_n(K^*)$



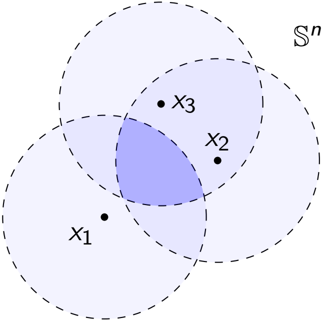
If $f_1, \dots, f_N$ have proper support,

$$I_\Psi(f_1^\star, \dots, f_N^\star) \geq I_\Psi(\|f_1\|_\infty \mathbb{1}_{C_1}, \dots, \|f_N\|_\infty \mathbb{1}_{C_N})$$

Random ball polyhedra

Theorem (H., Pivovarov)

Let $f_1, \ldots, f_N \in L^1(\mathbb{S}^n)$, $r > 0$



Random ball polyhedra

Theorem (H., Pivovarov)

Let $f_1, \ldots, f_N \in L^1(\mathbb{S}^n)$, $r > 0$

Intersection of random balls:

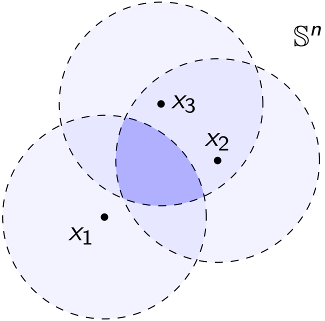
$$\Psi^r(x_1, \ldots, x_N) = \sigma_n\left(\bigcap_{i=1}^N C_r(x_i)\right)$$

$$I_{\Psi^r}(f_1 \ldots, f_N) \leq I_{\Psi^r}(f_1^\star \ldots, f_N^\star),$$

Union of random balls:

$$\Psi_r(x_1, \ldots, x_N) = \sigma_n\left(\bigcup_{i=1}^N C_r(x_i)\right)$$

$$I_{\Psi_r}(f_1 \ldots, f_N) \geq I_{\Psi_r}(f_1^\star \ldots, f_N^\star).$$



Random ball polyhedra

Theorem (H., Pivovarov)

Let $f_1, \ldots, f_N \in L^1(\mathbb{S}^n)$, $r > 0$

Intersection of random balls:

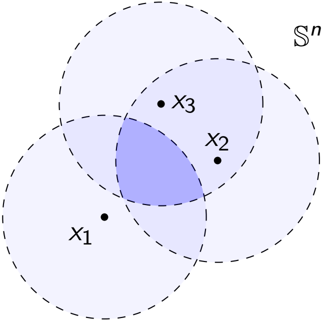
$$\Psi^r(x_1, \ldots, x_N) = \sigma_n\left(\bigcap_{i=1}^N C_r(x_i)\right)$$

$$I_{\Psi^r}(f_1 \ldots, f_N) \leq I_{\Psi^r}(f_1^\star \ldots, f_N^\star),$$

Union of random balls:

$$\Psi_r(x_1, \ldots, x_N) = \sigma_n\left(\bigcup_{i=1}^N C_r(x_i)\right)$$

$$I_{\Psi_r}(f_1 \ldots, f_N) \geq I_{\Psi_r}(f_1^\star \ldots, f_N^\star).$$



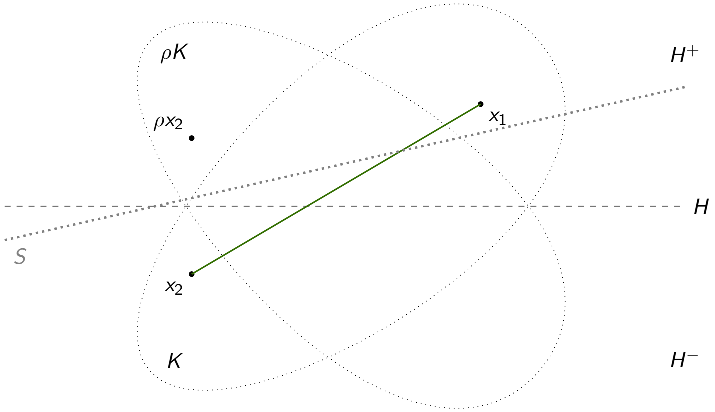
As $N \to \infty$,

- r-dual sets: $\sigma_n(K^r) \leq \sigma_n((C_K)^r)$ — Bezdek 2018
- Isoperimetric inequality: $\sigma_n(K_r) \geq \sigma_n((C_K)_r)$ — Lévy 1919

Sketch of proof

$x_1, \ldots, x_N \in K$:

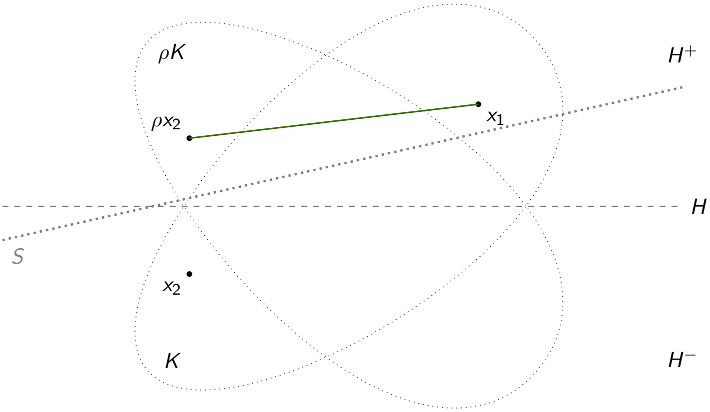
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\operatorname{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

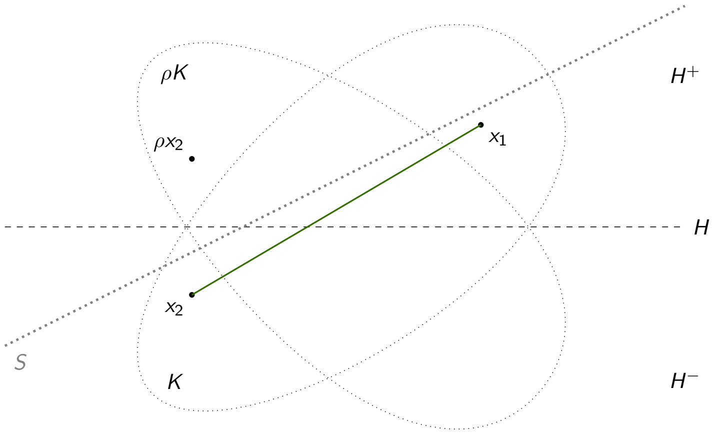
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

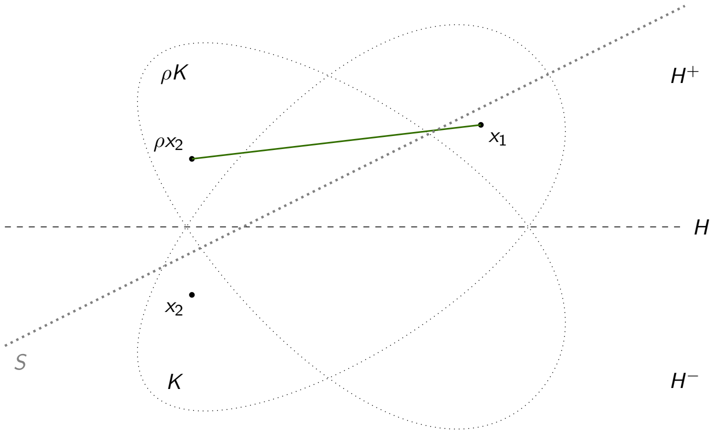
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S) \, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

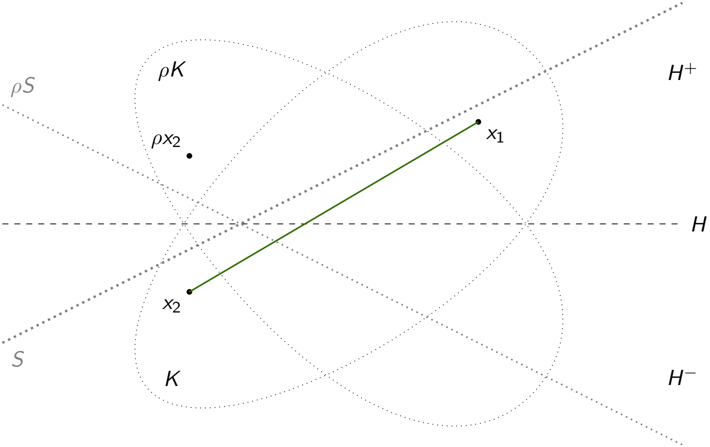
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

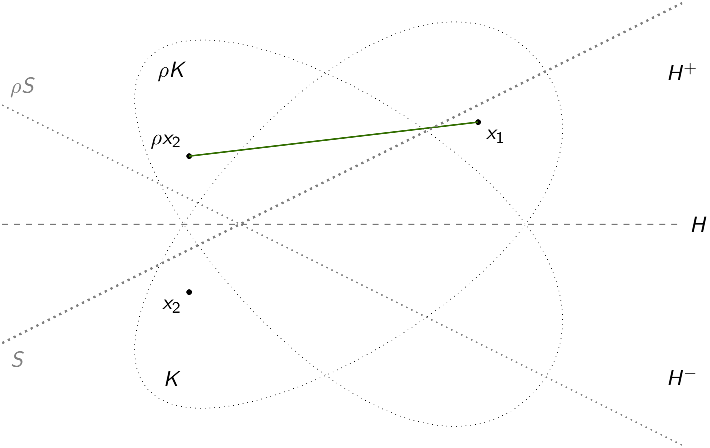
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

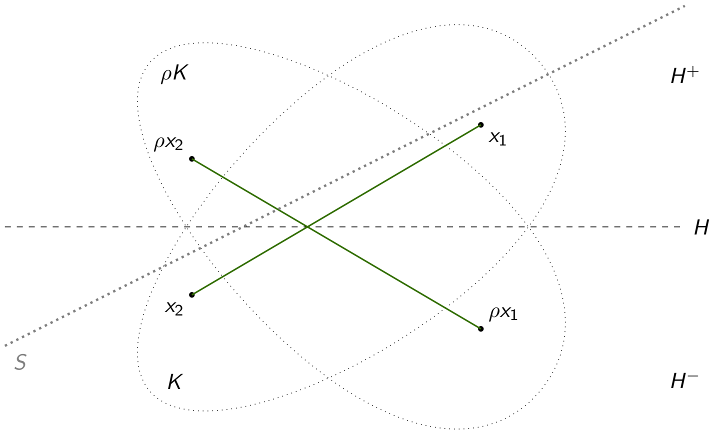
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

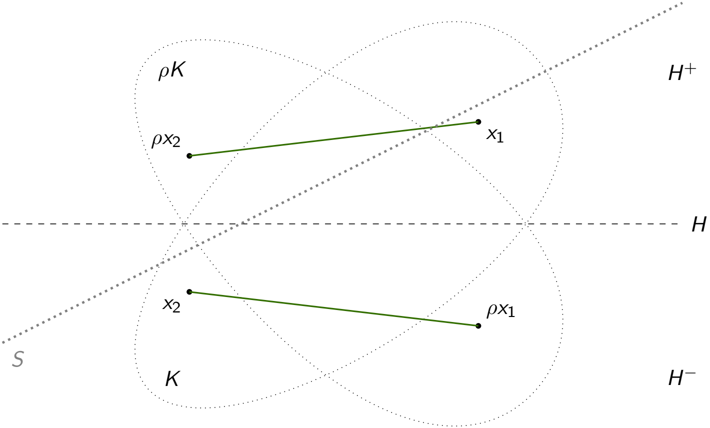
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \dots, x_N \in K$:

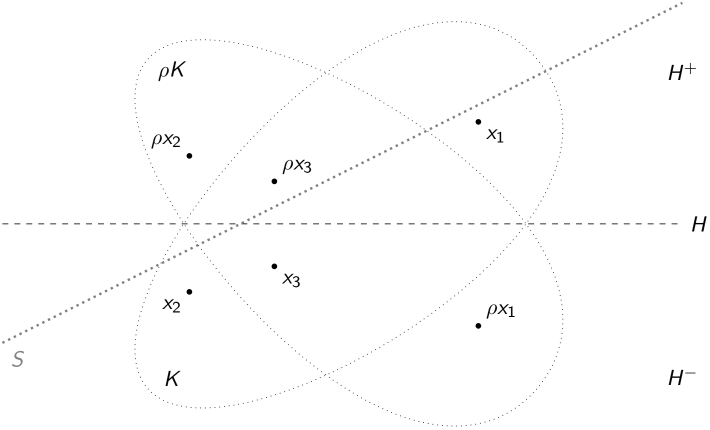
$$U_1(x_1, \dots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \dots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

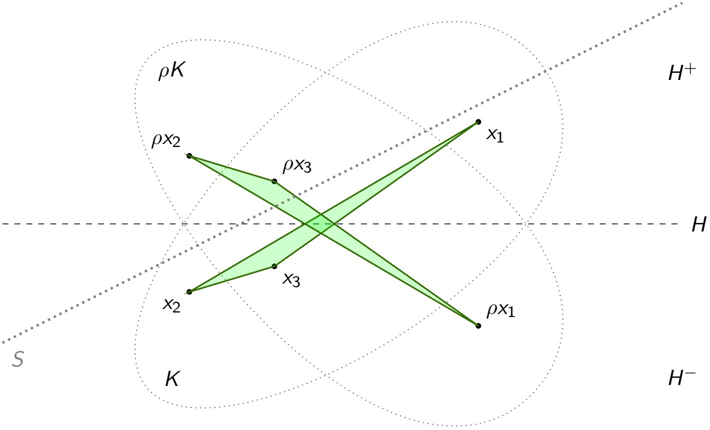
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

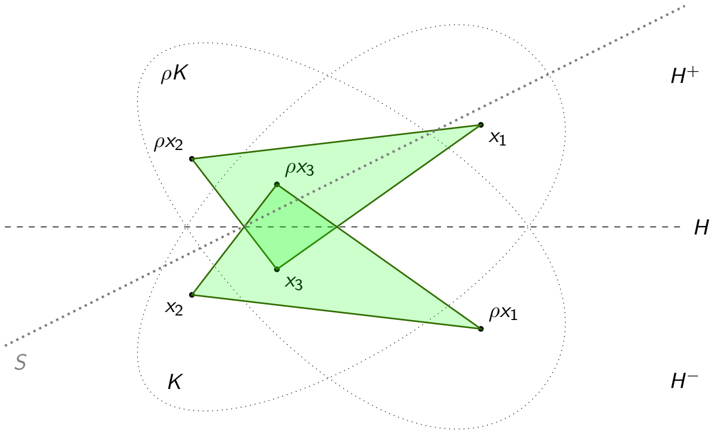
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

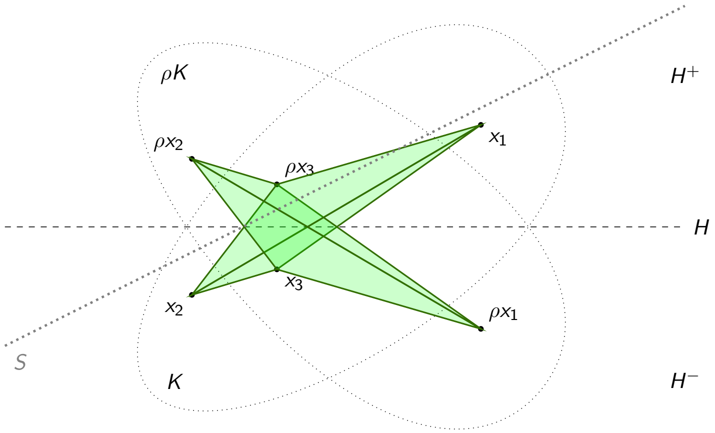
$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



Sketch of proof

$x_1, \ldots, x_N \in K$:

$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S) \, dS$$



Sketch of proof

$x_1, \dots, x_N \in K$:

$$U_1(x_1, \dots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \dots, x_N\} \cap S)\, dS$$