Bounds for the volume ratio of convex bodies

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The basics

Notation

- A *convex body* $K \subset \mathbb{R}^n$ is a compact convex set with nonempty interior.

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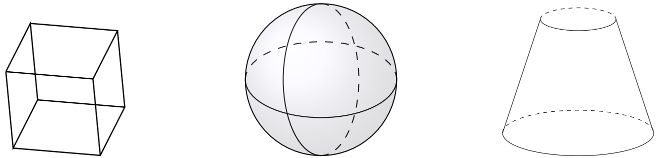


The basics

If $K \subset \mathbb{R}^n$ is centrally symmetric (i.e., $K = -K$) then K is the unit ball with respect to some norm $\|\cdot\|_K$ in $\mathbb{R}^n$.

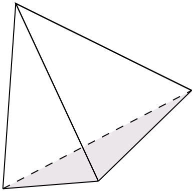
Notation

The normed space $(\mathbb{R}^n, \|\cdot\|_K)$ will be denoted by X_K.

The basics

Notation

- A simplex in $\mathbb{R}^n$ is always an n-simplex, the convex hull of $n+1$ affinely independent points.

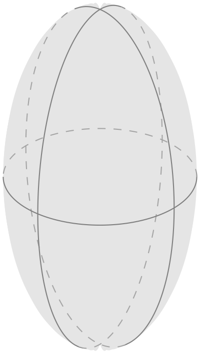


3-Simplex

The basics

Notation

- An ellipsoid in $\mathbb{R}^n$ is just an affine image of the euclidean unit ball.



The "standard volume ratio"

[ST80] Szarek, Stanisław, and Nicole Tomczak-Jaegermann.

On nearly Euclidean decomposition for some classes of Banach spaces. Compositio Mathematica 40.3 (1980): 367-385.

Definition

Given a convex body $K \subset \mathbb{R}^n$ its volume ratio is defined as

$$\mathrm{vr}(K) := \min \left(\frac{vol(K)}{vol(\mathcal{E})} \right)^{\frac{1}{n}},$$

where the minimum is taken over all ellipsoids $\mathcal{E}$ such that $\mathcal{E} \subset K$.

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Given a convex body $K \subset \mathbb{R}^n$ its volume ratio is defined as

$$\mathrm{vr}(K) := \min \left(\frac{vol(K)}{vol(T(B_2^n))} \right)^{\frac{1}{n}},$$

where the minimum is taken over all affine transformations T that $T(B_2^n) \subset K$.

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where the minimum is taken over all affine transformations T that $T(B_2^n) \subset K$.

In terms of measure this notion quantifies, in some sense, how well a convex body K can be approximated by an affine imagine of the euclidean ball.

Some elementary properties

The volume ratio is affinely invariant. That is

$$\mathrm{vr}(K) = \mathrm{vr}(TK),$$

for every affine transformation T.

John's position

Definition

A body $K \subset \mathbb{R}^n$ is in John's position if the euclidean ball B_2^n is the maximal volume ellipsoid inside K.

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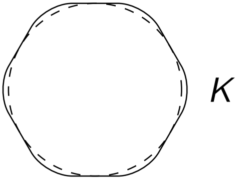

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A body $K \subset \mathbb{R}^n$ is in John's position if the euclidean ball B_2^n is the maximal volume ellipsoid inside K.

That is $B_2^n \subset K$ and

$$\mathrm{vr}(K) = \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2^n)}\right)^{1/n}.$$

For every body K, there is an affine transformation T such that $T(K)$ is in John's position.

Some remarks

[Joh48] John, Fritz. Extremum problems with inequalities as subsidiary conditions. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 187—204..

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If $K \subset \mathbb{R}^n$ is centrally symmetric in John's position (that is the euclidean ball B_2^n is the maximal volume ellipsoid inside K) then

$$B_2^n \subset K \subset \sqrt{n} B_2^n.$$

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In that case,

$$\mathrm{vr}(K) = \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_2^n)}\right)^{1/n} \leq \left(\frac{\mathrm{vol}(\sqrt{n}B_2^n)}{\mathrm{vol}(B_2^n)}\right)^{1/n} \leq \sqrt{n}.$$

In the non-symmetric case...

If $K \subset \mathbb{R}^n$ (not necessarily centrally symmetric) in John's position then

$$B_2^n \subset K \subset nB_2^n.$$

But we do still have the bound

$$\mathrm{vr}(K) \leq c\sqrt{n}.$$

since there is a reduction to the symmetric case that we will mention in a few minutes.

Extreme cases for the volume ratio

[Bal91] Ball, Keith Volume ratios and a reverse isoperimetric inequality. J. London Math. Soc. 44 (1991), 351-359.

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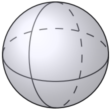
Result

$$\sup_{K \subset \mathbb{R}^n} \mathrm{vr}(K) = \mathrm{vr}(S),$$

where S is a simplex.

Extreme cases for the volume ratio

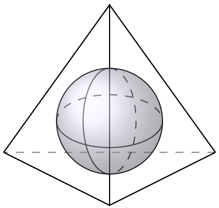
The Euclidean ball, B_2^n



Extreme cases for the volume ratio

The Euclidean ball, B_2^n

$B_2^n \subset \Delta_n$, the regular simplex, is in John's position.



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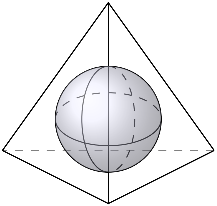

- $\mathrm{vol}(B_2^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$
- $\mathrm{vol}(\Delta_n) = \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!}$

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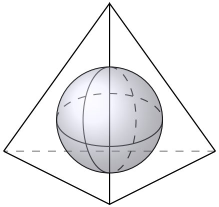

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Thus,

$$\mathrm{vr}(K) \leq \mathrm{vr}(\Delta_n) = \left(\frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)}{n! \pi^{n/2}} \right)^{\frac{1}{n}} \approx \sqrt{n}$$

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Result

$$\sup_{K=-K} \mathrm{vr}(K) = \mathrm{vr}(P),$$

where P is a parallelepiped.

Extreme cases for the volume ratio

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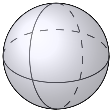
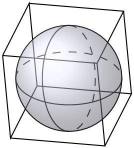


Extreme cases for the volume ratio

The Euclidean ball, B_2^n

$B_2^n \subset B_\infty^n$, the unit ball of ℓ_∞^n, is in John's position.



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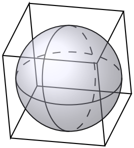

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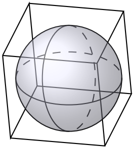

- $\mathrm{vol}(B_2^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$
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Thus if $K = -K$,

$$\mathrm{vr}(K) \leq \mathrm{vr}(B_\infty^n) = \left(\frac{2^n\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}\right)^{\frac{1}{n}} \approx \sqrt{n}$$

General Volume ratio

Definition

Given two convex bodies $K, L \subset \mathbb{R}^n$ the volume ratio between K and L is

$$\mathrm{vr}(K, L) := \min \left(\frac{vol(K)}{vol(T(L))} \right)^{\frac{1}{n}},$$

where the minimum is taken over all affine transformations T such that $T(L) \subset K$.

[GLMP04] Y. Gordon, A.E. Litvak, M. Meyer, and A. Pajor

John's Decomposition in the General Case and Applications *J. Differential Geom.*, 2004.

Remarks

- If $L = B_2^n$ we recover the standard notion of volume ratio. That is,

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- $\mathrm{vr}(K, L) \leq \mathrm{vr}(K, Z)\mathrm{vr}(Z, L)$.

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for every affine transformation T, S.
- $\mathrm{vr}(K, L) \leq \mathrm{vr}(K, Z)\mathrm{vr}(Z, L)$.
- $\mathrm{vr}(K, L) \approx \mathrm{vr}(L^\circ, K^\circ)$.

General upper bounds for the volume ratio

Giannopoulos-Hartzoulaki (2002)

For every pair of convex bodies $K, L \subset \mathbb{R}^n$

$$\mathrm{vr}(K, L) \leq C\sqrt{n}\log(n).$$

[GH02] A Giannopoulos, M Hartzoulaki On the volume ratio of two convex bodies. Bulletin of the London Mathematical Society 34.6 (2002): 703-707.

Lower bounds

Khrabrov (2001)

Given a convex body K there is Z such that

$$\mathrm{vr}(K, Z) \geq C\sqrt{\frac{n}{\log\log(n)}}.$$

[Khr01] Alexander Igorevich Khrabrov. Generalized volume ratios and the Banach–Mazur distance. *Mathematical Notes*, 2001.

The largest volume ratio

Given a convex body $K \subset \mathbb{R}^n$ we define its largest volume ratio as

$$\mathrm{lvr}(K) := \sup_{L \subset \mathbb{R}^n} vr(K, L).$$

Remarks

- For every convex body K we have:

$$\sqrt{\frac{n}{\log\log(n)}} \preceq \mathrm{lvr}(K) \preceq \sqrt{n}\log(n).$$

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Remarks

- For every convex body K we have:

$$\sqrt{\frac{n}{\log\log(n)}} \preceq \mathrm{lvr}(K) \preceq \sqrt{n}\log(n).$$

- For many bodies $K \subset \mathbb{R}^n$, $\mathrm{lvr}(K) \approx \sqrt{n}$.

Question:

Can we improve the general bounds?

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Question:

Can we show sharp asymptotic estimates for certain classes of convex bodies?

Lower estimates...

Recall... For every convex body K, the best known general bounds for the largest volume ratio are

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G., Merzbacher, Pinasco, 2019+

For every convex body K we have:

$$\sqrt{n} \preceq \operatorname{lvr}(K)$$

Reduction to the symmetric case

Rogers-Shephard Inequality

Given a convex body K,

$$vol(K-K)^{\frac{1}{n}} \leq 4vol(K)^{\frac{1}{n}}.$$

Reduction to the symmetric case

Rogers-Shephard Inequality

Given a convex body K,

$$vol(K-K)^{\frac{1}{n}} \leq 4vol(K)^{\frac{1}{n}}.$$

A body which contains the origin can be approximated by outside by a symmetric body with essentially the same volume.

$$\mathrm{vr}(K-K,K) \leq 4.$$

Then,

$$\mathrm{vr}(K-K,Z) \leq \mathrm{vr}(K-K,K)\mathrm{vr}(K,Z) \leq 4\mathrm{vr}(K,Z).$$

Assumption

Since there is always a reduction (considering either $K - K$ or $K \cap -K$ depending on the case) we will assume for simplicity that all bodies involved are centrally symmetric.

If K and Z are centrally symmetric

$$\mathrm{vr}(K,Z) = \frac{vol(K)^{\frac{1}{n}}}{vol(Z)^{\frac{1}{n}}} \cdot \inf\left\{\|T : X_Z \to X_K\| : \det(T) = 1\right\}$$

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Idea:

Given a fixed centrally symmetric body $K \subset \mathbb{R}^n$, find Z such that

- for every $T \in SL_n(\mathbb{R})$ the norm $\|T : X_Z \to X_K\|$ is big,
- the measure $vol(Z)^{\frac{1}{n}}$ is small.

But how???

How?

The probabilistic method...

... is a nonconstructive method for proving the existence of a prescribed kind of mathematical object.

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... is a nonconstructive method for proving the existence of a prescribed kind of mathematical object.

It works by showing that if one randomly chooses objects from a specified class, the probability that the result is of the prescribed kind is strictly greater than zero.

Random polytopes on the sphere (Gluskin's polytopes)

Given $m \in \mathbb{N}$, we consider the set

$$\mathcal{A}_m := \{\text{absconv}\{e_1 \dots e_n, f_1 \dots, f_m\} : f_k \in \mathbb{S}^{n-1}\}.$$

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Note that we have the following mapping

$$\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1} \to \mathcal{A}_m,$$

given by $(f_1, \dots, f_m) \mapsto \mathrm{absconv}\{e_1 \dots e_n, f_1 \dots, f_m\}$.

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This induces a measure ν in $\mathcal{A}_m$: the push-forward of the product measure $\mu_n \times \mu_n \times \cdots \times \mu_n$, where μ_n is the probability surface measure on $\mathbb{S}^{n-1}$.

Carl/Pajor - Gluskin:

If $Z \in \mathcal{A}_m$, then

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In particular, if $m \sim n$ then $vol(Z)^{\frac{1}{n}} \preceq \frac{1}{n}$.

$$\mathcal{B}_m = \{Z \in \mathcal{A}_m : \exists T \in SL_n(\mathbb{R}) \text{ with } \|T : X_Z \to X_K\| \leq \frac{\beta}{\sqrt{n} vol(K)^{\frac{1}{n}}}\}.$$

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G., Merzbacher, Pinasco: refinement of Khrabrov's result

Let $K \subset \mathbb{R}^n$ be a centrally symmetric body. Then,

$$\nu(\mathcal{B}_m) \leq C^{n^2}(\|id : \ell_2^n \to X_K\| \sqrt{n} vol(K)^{\frac{1}{n}})^{n^2} \beta^{mn}.$$

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Suppose that for a given convex body K we have that $\rho(K) := \|id : \ell_2^n \to X_K\|\sqrt{n}vol(K)^{\frac{1}{n}}$ is bounded by an absolute constant. Thus, $\nu(\mathcal{B}_m) \leq D^{n^2}\beta^{mn}$... By picking $m \sim n$ and β small enough we have $\nu(\mathcal{B}_m) < 1$.

$$\mathcal{B}_m = \{Z \in \mathcal{A}_m : \exists T \in SL_n(\mathbb{R}) \text{ with } \|T : X_Z \to X_K\| \leq \frac{\beta}{\sqrt{n} vol(K)^{\frac{1}{n}}}\}.$$

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Therefore, the complement is non-empty.

Thus, there is some $Z \in \mathcal{A}_{cn}$ (in particular $vol(Z)^{\frac{1}{n}} \sim \frac{1}{n}$) that is not in $\mathcal{B}_{cn}$. Note that in that case for every $T \in SL_n(\mathbb{R})$ we have

$$\|T : X_Z \to X_K\| \geq \frac{\beta}{\sqrt{n} vol(K)^{\frac{1}{n}}}.$$

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$$\|T : X_Z \to X_K\| \geq \frac{\beta}{\sqrt{n} vol(K)^{\frac{1}{n}}}.$$

Then,

$$\begin{aligned} \mathrm{vr}(K, Z) &= \frac{vol(K)^{\frac{1}{n}}}{vol(Z)^{\frac{1}{n}}} \cdot \inf \{\|T : X_Z \to X_K\| : \det(T) = 1\} \\ &\geq n vol(K)^{\frac{1}{n}} \cdot \frac{\beta}{\sqrt{n} vol(K)^{\frac{1}{n}}} \approx \sqrt{n} \end{aligned}$$

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$$\nu(\mathcal{B}_m) \leq C^{n^2}(\|id : \ell_2^n \to X_K\|\sqrt{n}vol(K)^{\frac{1}{n}})^{n^2}\beta^{mn}.$$

How to proceed in general if $\rho(K) := \|id : \ell_2^n \to X_K\|\sqrt{n}vol(K)^{\frac{1}{n}}$ is not bounded by an absolute constant? Approximate the body K with a one that fulfills this hypothesis, without losing volume (the theory of isotropic convex bodies is involved).

How to approximate the body with a one with bounded ρ

Given a centrally symmetric body $K \subset \mathbb{R}^n$ there is W such that $\mathrm{vr}(W, K) \leq C_1$ and

$$\rho(W) = \|id : \ell_2^n \to X_W\| \sqrt{n} vol(W)^{\frac{1}{n}} \leq C_2$$

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- Suppose that K° is in isotropic position (so it has volume one). By a deep result of Paouris (2006) most of the mass of K° concentrates inside $\sqrt{n} B_2^n$.
- Take $W^\circ := K^\circ \cap \sqrt{n} B_2^n$. Observe that $vol(W^\circ)^{\frac{1}{n}} \sim 1$ so, $vol(W)^{\frac{1}{n}} \sim \frac{1}{n}$. On the other hand,

$$\mathrm{vr}(W, K) \sim \mathrm{vr}(K^\circ, W^\circ) \le C.$$

How to approximate the body with a one with bounded ρ

Given a centrally symmetric body $K \subset \mathbb{R}^n$ there is W such that $\mathrm{vr}(W, K) \leq C_1$ and

$$\rho(W) = \|id : \ell_2^n \to X_W\| \sqrt{n} vol(W)^{\frac{1}{n}} \leq C_2$$

- Suppose that K° is in isotropic position (so it has volume one). By a deep result of Paouris (2006) most of the mass of K° concentrates inside $\sqrt{n}B_2^n$.
- Take $W^\circ := K^\circ \cap \sqrt{n}B_2^n$. Observe that $vol(W^\circ)^{\frac{1}{n}} \sim 1$ so, $vol(W)^{\frac{1}{n}} \sim \frac{1}{n}$. On the other hand,

$$\mathrm{vr}(W, K) \sim \mathrm{vr}(K^\circ, W^\circ) \leq C.$$

- By construction, $W^\circ \subset \sqrt{n}B_2^n$ then $B_2^n \subset \sqrt{n}W$ (which implies that $\|id : \ell_2^n \to X_W\| \leq \sqrt{n}$).

Upper estimates...

Largest volume ratio for the cube

Lemma (Dvoretzky-Rogers)

Let $L \subset \mathbb{R}^n$ be a convex body in John's position, there are $y_1 \dots y_n \in Bd(B_2^n) \cap Bd(L)$ such that

$$\|P_{span\{y_1 \dots y_{i-1}\}^\perp}(y_i)\| \geq \left(\frac{n-i+1}{n}\right)^{\frac{1}{2}}.$$

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$\det|y_1 \dots y_n| = \|y_1\| \prod_{i=2}^n \|P_{span\{y_1 \dots y_{i-1}\}^\perp}(y_i)\| \geq \left(\frac{n^n}{n!}\right)^{\frac{1}{2}}$

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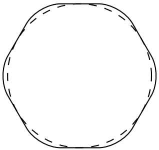

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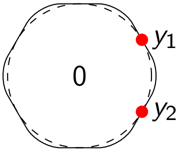

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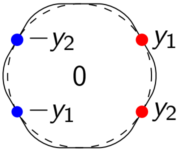

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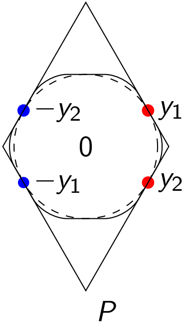

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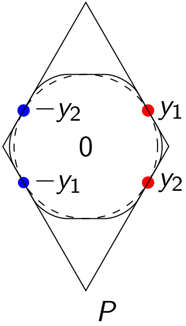

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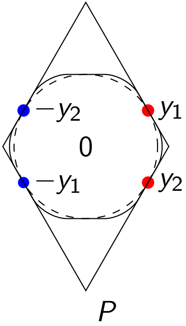

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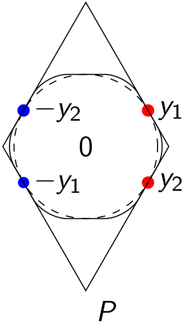

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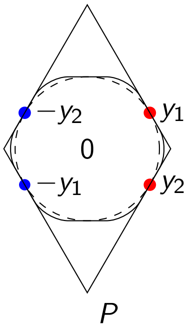

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$$\mathrm{lvr}(B_\infty^n) \sim \sqrt{n}.$$

Unconditional bodies

Definition

A convex body $K \subset \mathbb{R}^n$ is unconditional if $(x_1, x_2, \ldots, x_n) \in L$ then $(\varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_n x_n) \in L$ for all $\varepsilon_i \in \{-1, 1\}$.

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Example

- unconditional: B_p^n for all $1 \leq p \leq \infty$,
- not unconditional: $B_{\mathcal{L}(\ell_2^n)} \subset \mathbb{R}^{n^2}$.

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Bobkov-Nazarov (2003)

There is a constant $c > 0$ (independent of the dimension) such that for every unconditional isotropic convex body $K \subset \mathbb{R}^n$ then

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Indeed,

$$\mathrm{vr}(K, L) \leq \mathrm{vr}(K, B_\infty^n)\mathrm{vr}(B_\infty^n, L) \leq C\sqrt{n}.$$

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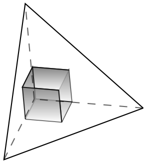

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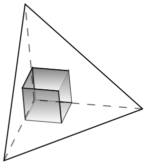

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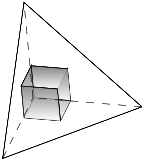

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Applying Stirling's formula:

$$\mathrm{vr}(S, B_\infty^n) = \mathrm{vr}(S, C) \leq \left(\frac{\mathrm{vol}(S)}{\mathrm{vol}(C)}\right)^{\frac{1}{n}} \approx 1$$

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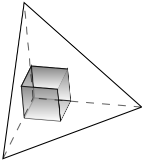

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Therefore

$$\text{lvr}(S) \sim \sqrt{n}.$$

Schatten Classes

Given $A \in M_d(\mathbb{R})$, consider $s(A) = (s_1(A), \dots, s_d(A))$ the sequence of eigenvalues of $(AA^*)^{\frac{1}{2}}$. We define the p-Schatten norm on $\mathbb{R}^{d^2}$ as

$$\sigma_p(A) = \|S(A)\|_{\ell_p^d} = (tr|A|^p)^{1/p},$$

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We denote $B_{S_p^d}$ unit ball of the p-Schatten class in $\mathbb{R}^{d^2}$.

G., Merzbacher, Pinasco

For every $1 \leq p \leq \infty$, the largest volume ratio of the unit ball of the p-Schatten class (which is a set in $\mathbb{R}^{d^2}$) behaves as

$$\mathrm{lvr}(B_{S_p^d}) \sim d.$$

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How? We give a very careful look at the proofs of the general upper inequalities. Again, all relies on the probabilistic method!!!!!!

Given $T : X_L \to X_K$ we have $T(L) \subset \|T\| K$ and so

$$\mathrm{vr}(K, L) \leq \frac{\|T\| vol(K)^{\frac{1}{n}}}{(\det T)^{\frac{1}{n}} vol(L)^{\frac{1}{n}}}.$$

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Chevet's inequality

If we denote $\mathcal{O}(n)$ the orthogonal group endowed with the Haar probability measure, then

$$\mathbb{E}\|T : X_L \to X_K\| \leq \frac{C}{\sqrt{n}}(\ell(K)\|id : L \to \ell_2^n\| + \|id : \ell_2^n \to X_K\|\ell(L^\circ)).$$

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$$\mathrm{vr}(K, L) \leq \alpha(K, L)\frac{vol(K)^{\frac{1}{n}}}{vol(L)^{\frac{1}{n}}}.$$

Idea: Since the volume ratio is affinely invariant we look for a position of L such that

- $\|id : L \to \ell_2^n\|$ and $\ell(L^\circ)$ are "not so big".
- $vol(L)^{\frac{1}{n}}$ is "not so small".

Proposition:

For every centrally symmetric convex body $K \subset \mathbb{R}^n$,

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Similarly we can show that

For every $1 \leq p \leq \infty$:

$$\text{lvr}(\otimes_\pi^m \ell_p^n) \sim \text{lvr}(\otimes_\epsilon^m \ell_p^n) \sim n^{m/2}.$$

Recent resuts

Theorem:

Given $\delta > 0$ there is a constant $d := d(\delta) > 0$ with the following property:
For each convex body $K \subset \mathbb{R}^n$ and $\delta n \leq k \leq n$, there is a centrally symmetric body $Z \subset \mathbb{R}^n$ such that

$$\mathrm{vr}(QK, QZ) \geq d\sqrt{\frac{k}{\log\log k}},$$

for every orthogonal projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k.

Thank you very much for your attention!