The Weyl principle in Finsler geometry

joint work (in progress) with Thomas Wannerer

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Convex, discrete and integral geometry

Friedrich Schiller University, Jena, September 2019

Euclidean space

$B^n \in \mathcal{K}(\mathbb{R}^n)$ is the Euclidean unit ball, $\omega_n = |B^n|$.

For $K \in \mathcal{K}(\mathbb{R}^n)$, $\mu_k(K)$, $k = 0, \ldots, n$ are its **intrinsic volumes**.

They can be defined in several ways:

- **Steiner formula**: $\mathrm{vol}(K + \epsilon B^n) = \sum_{k=0}^n \omega_{n-k}\mu_k(K)\epsilon^{n-k}$.
- **Crofton formulas.**

$$\mu_k(K) = \int_{\mathrm{AGr}_{n-k}(\mathbb{R}^n)} \chi(K \cap E) d\sigma_{n-k}(E)$$

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Valuations on manifolds

- Let M^n be a smooth oriented manifold with some Riemannian structure, SM the unit sphere bundle
- $\mathcal{P}(M)$ will be the set of compact differentiable polyhedra.

Definition

For $K \in \mathcal{P}(M)$, its normal cycle is $nc(K) \subset SM$. $(x, v) \in nc(K) \iff \langle v, \dot\gamma \rangle \leq 0$ for all curves $\gamma \subset K$ s.t. $\gamma(0) = x$. $nc(K) \subset SM$ is a Lipschitz submanifold of dimension $n - 1$.

Definition (Alesker, Fu)

A pair $\mu \in \Omega^n(M)$, $\omega \in \Omega^{n-1}(SM)$ defines a smooth valuation $\phi \in \mathcal{V}^\infty(M)$, given by $\phi : \mathcal{P}(M) \to \mathbb{R}$, $\phi(K) = \int\limits_K \mu + \int\limits_{nc(K)} \omega$.

Theorem (Alesker)

- *There is a product* $\mathcal{V}^\infty(M) \otimes \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$.
- *If* $e : M \hookrightarrow N$ *is an embedding, the restriction* $e^* : \mathcal{V}^\infty(N) \to \mathcal{V}^\infty(M)$ *respects the product. We write* $e^*\phi = \phi|_M$.

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Riemannian Lipschitz-Killing curvatures

Theorem (Weyl)

Let (M^n, g) be a closed Riemannian manifold. Embed isometrically $M \subset \mathbb{R}^N$. Let M_ϵ denote the ϵ-extension. Then $\mathrm{vol}_N(M_\epsilon) = \sum_{k=0}^n \omega_{n-k}\mu_k(M)\epsilon^{N-k}$ for small ϵ.
Remarkably, $\mu_k(M)$ only depends on (M, g).

Theorem (Weyl, Chern, ...)

There are natural k-homogeneous assignments μ_k, $k = 0, 1, \ldots$ that assign to a Riemannian manifold a valuation:

$$\mu_k^{M,g} \in \mathcal{V}^\infty(M), \qquad \mu_k^{M,\lambda^2 g} = \lambda^k \mu_k^{M,g}.$$

If $e : M \hookrightarrow N$ is an isometric embedding, $e^*\mu_k^N = \mu_k^M$.

Examples: For $X \subset M^n$: $\mu_0(X) = \chi(X)$, $\mu_n(X) = \mathrm{vol}_n(X)$, $\mu_{n-1}(X) = \frac{1}{2}\mathrm{vol}_{n-1}(\partial X)$.

The Weyl principle - loosely defined as the existence in a category of natural valuations that are invariant under embeddings - has been extended to Kähler, semi-Riemannian, and contact manifolds. Conjectured extension to Alexandrov spaces.

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Volumes in Banach space

Let (V, F) be a normed space with unit ball B_F, henceforth assumed smooth and strictly convex.

How to measure volume in V? More than one possibility:

Definition (Busemann volume)

Set $\mathrm{vol}^{Bu}_{V,F}(B_F) = \omega_n$. This is just the Hausdorff measure.

From the symplectic and integral-geometric perspectives, the following definition appears more natural.

Definition (Holmes-Thompson volume)

Set $\mathrm{vol}^{HT}_{V,F}(B_F) = \frac{1}{\omega_n}|B_F \times B_F^o|$.

- If $E \subset V$ is a linear subspace, it inherits a norm $F|_E$ and so a Lebesgue measure $\mathrm{vol}^{HT}_{E,F} \in \mathrm{Dens}(E)$.
- If $X^k \in \mathcal{P}(V)$ is compact, we can compute $\mathrm{vol}^{HT}_k(X) := \int_X d\,\mathrm{vol}^{HT}_{T_xX,F}$.

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The Holmes-Thompson intrinsic volumes

Theorem (Schneider-Wieacker, Alvarez-Paiva – Fernandes, Bernig)

Let $(\mathbb{R}^n, F)$ be a normed space. There are for $0 \leq k \leq n$ translation-invariant valuations $\mu_k^F \in \mathcal{V}^\infty(\mathbb{R}^n)$, k-homogeneous in F, such that for all $E \subset \mathbb{R}^n$ with $\dim E = k$, $\mu_k^F|_E = \mathrm{vol}_{E,F}^{HT}$.

$\Rightarrow$ If $U \subset V$ is isometric, $\mu_k^V|_U = \mu_k^U$.

Construction:

- Using the Alesker-Fourier transform:

$$\mu_k^F = c_{n,k}\mathbb{F}V(\bullet[n-k], B_F^o[k])$$

- Using Crofton formulas (Alvarez-Paiva – Fernandes):

$$\mu_k^F(K) = \int_{\mathrm{AGr}_{n-k}(\mathbb{R}^n)} \chi(K \cap E) d\Phi_{n-k}(E)$$

Here $\Phi_{n-k} \in \mathcal{M}(\mathrm{AGr}_{n-k}(\mathbb{R}^n))^{tr}$ depends explicitly on F. In particular, Φ_{n-1} can be identified with the inverse cosine transform of $F \in C^\infty(S^{n-1})$.

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Remark

Note that the Holmes-Thompson intrinsic volumes $\mu_k^F(K)$ are **not** the coefficients of $\mathrm{vol}_{HT}(K+\epsilon B_F)$.

Uniqueness (in stance of Hadwiger's characterization):

- μ_k^F is uniquely determined by the condition $\mu_k^F|_E = \mathrm{vol}_{E,F}^{HT}$ $\forall E \in \mathrm{Gr}_k(\mathbb{R}^n)$, by Klain's theorem.
- Alternatively: μ_1^F is uniquely determined by $\mu_1([0,x]) = F(x)$ for all $x \in \mathbb{R}^n$. For $k>1$, $\mu_k^F = c_{n,k}(\mu_1^F)^k$.

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Finsler manifolds

Definition

A Finsler manifold (M, f) is a smooth manifold M with $f \in C^\infty(TX \setminus \underline{0})$ a strictly convex norm on every tangent space.

Question (Fu)

Given an isometric embedding $e : (M, f) \hookrightarrow (V, F)$ of a Finsler manifold in a normed space, does $e^*\mu_k^F = \mu_k^F|_{e(M)}$ depend on (M, f) alone?

The answer is trivially YES for dim $M \leq 2$: for $X^2 \subset M^2$,

$$\mu_0^F(X) = \chi(X), \quad \mu_1^F(X) = \frac{1}{2}\mathrm{Length}_F(\partial X), \quad \mu_2^F(X) = \mathrm{Area}_F^{HT}(X)$$

Answer (F-Wannerer)

The Weyl principle does not hold in the Finsler category:
$\exists$ a Finsler manifold M^3 and two isometric embeddings $e_j : M \hookrightarrow (\mathbb{R}^7, F_j)$, $j = 1, 2$, such that $e_1^*\mu_1^{F_1} \neq e_2^*\mu_1^{F_2}$.

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Given an isometric embedding $e : (M, f) \hookrightarrow (V, F)$ of a Finsler manifold in a normed space, does $e^*\mu_k^F = \mu_k^F|_{e(M)}$ depend on (M, f) alone?

The answer is trivially YES for $\dim M \leq 2$: for $X^2 \subset M^2$,

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The Weyl principle does not hold in the Finsler category: $\exists$ a Finsler manifold M^3 and two isometric embeddings $e_j : M \hookrightarrow (\mathbb{R}^7, F_j)$, $j = 1, 2$, such that $e_1^*\mu_1^{F_1} \neq e_2^*\mu_1^{F_2}$.

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Assume $M^m \subset V^n$ is a submanifold in a linear space. It satisfies the global weak Weyl principle if whenever F_1, F_2 are two norms on V such that $F_1|_M = F_2|_M$, then also $\mu_k^{F_1}|_M = \mu_k^{F_2}|_M$ for all k.

It is more natural to consider this property locally.

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We say $M \subset V$ satisfies the local weak Weyl principle if GWWP holds for all open subsets $U \subset M$.

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Assume $\dim V \leq 2 \dim M$. *Then* $M \subset V$ *satisfies LWWP.*

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By a theorem of Bernig-Bröcker, there is a natural linear inclusion $(C, T) : \mathcal{V}^\infty(M) \hookrightarrow C^\infty(M) \times \Omega^n(SM)$.

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Consider (V^n, F), $\mu_1^F \in \mathcal{V}^\infty(V)$, $F \in C^\infty(S^{n-1})$.

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Preparation 2 - the restriction

- Let $\Phi M = \{(p; u_j)_{j=0}^{m-1}\}$ be the orthonormal frame bundle over M, $u_0 : \Phi M \to SM$. Identify $\Omega(SM) \equiv u_0^* \Omega(SM) \subset \Omega(\Phi M)$.
- Let $\theta_j = \langle u_j, \bullet \rangle \in \Omega^1(\Phi M)$ be the solder forms, and $\omega_{i,j} \in \Omega^1(\Phi M)$ the corresponding connection forms.
- Let P_{m-1}^k be the set of increasing functions $\sigma : \{1, \ldots, k\} \to \{1, \ldots, m-1\}$.

$$T(e^* \mu_1^F) = \theta_0 \wedge \sum_{k=0}^{m-1} \sum_{\sigma \in P_{m-1}^k} \sum_{\tau \in P_{m-1}^{m-1-k}} A_{\sigma\tau}^k \bigwedge_{j=1}^{k} \theta_{\sigma(j)} \wedge \bigwedge_{j=1}^{m-1-k} \omega_{\tau(j),0}.$$

The coefficients are given by

$$A_{\sigma\tau}^k(p; u_j) = \int_0^{\pi/2} \int_{S(T_p^\perp M)} \cos^{m-k}\phi \sin^{n-m-1+k}\phi \cdot$$
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$\theta : SM \to S^{n-1}$ is the obvious map $\theta(p,u) = u$.

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$$A^0|_{S_pM} = \mathcal{C}_m^{-1}(F|_{S_pM})$$

In particular, A^0 only depends on $F|_{\theta(SM)}$.

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Integral evaluation: general case

For $F \in C^\infty(S^{n-1})$, $u \in S^{n-1}$, $N \in S(T_u^\perp S^{n-1})$, $\lambda \in \mathbb{R}$, $\tilde{N} = \lambda N$ write

$$\frac{\partial^k}{\partial \tilde{N}^k} F(u) = \lambda^k \frac{\partial^k}{\partial N^k} F(u).$$

Proposition - a stability estimate

The coefficients $A^k_{\sigma\tau}$ at $p \in M$ are determined by the restriction to S_pM of all derivatives $\frac{\partial^i}{\partial^i \tilde{N}} F$, $i \le k$, for all vectors $\tilde{N} \in \mathrm{Image}(h_p)$. Moreover, there are constants $L = L(n,m)$ and $c = c(n,m)$ such that

$$\|A^k_{\sigma\tau}\|_{C^0(S_pM)} \le c \sup_{v_1, v_2 \in S_pM} \sum_{j=0}^{k} \|\frac{\partial^j}{\partial [h_p(v_1, v_2)]^j} F\|_{C^L(S_pM)}$$

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Back to geometry

- Key observation: For $u \in S_pM$, $\tilde{N} \in T_p^\perp M$, $\frac{\partial^j}{\partial \tilde{N}^j} F(u)$ is determined by $F|_{\theta(SM)}$ if $\tilde{N} = h_p(u, v)$ for some $v \in T_pM$.
  Letting $u(t)$ be the parallel transport of u along $p(t)$, $p'(0) = v$,

$$\frac{\partial F}{\partial [h_p(u,v)]}(u) = \frac{d}{dt}\bigg|_{t=0} F(u(t))$$

- For generic M,

$$\dim \mathrm{Image}(h_p(u, \bullet)) = \min(m, n-m)$$

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The End

Thanks for listening!