Brunn-Minkowski type inequalities and conjectures

Károly Böröczky
Alfréd Rényi Institute of Mathematics and CEU

Jena, September, 2019

Brunn-Minkowski inequality

K, C convex bodies in $\mathbb{R}^n$, $\alpha, \beta > 0$

$$\begin{aligned} \alpha K + \beta C &= \{\alpha x + \beta y : x \in K,\ y \in C\} \\ &= \{x \in \mathbb{R}^n : \langle u, x \rangle \leq \alpha h_K(u) + \beta h_C(u)\ \forall u \in S^{n-1}\} \end{aligned}$$

Brunn-Minkowski inequality $\alpha, \beta > 0$

$$V(\alpha\, K + \beta\, C)^{\frac{1}{n}} \geq \alpha\, V(K)^{\frac{1}{n}} + \beta\, V(C)^{\frac{1}{n}}$$

with equality iff K and C are homothetic ($K = \gamma C + x$, $\gamma > 0$).

Equivalent form $\lambda \in (0, 1)$

$$V((1-\lambda)\, K + \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$

Optimal transportation to prove B-M inequality

$V(K) = V(C) = 1$, K, C convex bodies in $\mathbb{R}^n$
Caffarelli, Brenier
$\exists\ C^\infty$ convex $\varphi : \mathrm{int} K \to \mathbb{R}$ such that $T = \nabla\varphi : \mathrm{int} K \to \mathrm{int} C$
bijective & $\det \nabla T = \det \nabla^2 \varphi = 1$

Gromov's argument for Brunn-Minkowski (appendix to Milman-Schechtman)
$\lambda \in (0,1)$, $y = (1-\lambda)x + \lambda T(x) \in (1-\lambda)K + \lambda C \Longrightarrow$
$dy = \det[(1-\lambda)\mathrm{I}_n + \lambda \nabla T(x)]\, dx$

$$V((1-\lambda)K + \lambda C) \geq \int_K \det[(1-\lambda)\mathrm{I}_n + \lambda \nabla T(x)]\, dx \geq \int_K 1\, dx = 1$$

$\det[(1-\lambda)A + \lambda B] \geq (\det A)^{1-\lambda}(\det B)^{\lambda}$ for positive definite A, B

Figalli, Maggi, Pratelli - stability of Brunn-Minkowski (strongest version by Kolesnikov, Milman)

Surface area measure, Minkowski's first inequality

S_K - surface area measure on S^{n-1} of a convex body K in $\mathbb{R}^n$

- ∂K is $C^2_+ \implies$

$$dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}$$

$\kappa(u) =$Gaussian curvature at $x \in \partial K$ where u is normal.

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i).$$

Minkowski's first inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K,$$

with equality iff K and C are translates.

Minkowski problem - characterize S_K

Given Borel measure μ on S^{n-1} with $\int_{S^{n-1}} u\, d\mu(u) = o$=origin, to solve the Minkowski problem finding K with $\mu = S_K$,

- Minimize $\int_{S^{n-1}} h_C\, d\mu$ under the condition $V(C) = 1$
- Uniqueness up to translation comes from uniqueness in the Minkowski inequality

Monge-Ampere type differential equation on S^{n-1}:

$$\det(\nabla^2 h + h\, I_{n-1}) = \kappa^{-1}$$

where $h(u) = h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ support function.

Curvature function For any convex body K,

$$f_K(u) = \det(\nabla^2 h_K(u) + h_K(u)\, I_{n-1})$$

for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$

Decomposition of Surface area measure

Lebesgue's decomposition of S_K for a convex body K
$S_K = S_K^a + S_K^s$ where S_K^s singular

$$dS_K^a = f_K \, d\mathcal{H}^{n-1}$$

Minkowski problem for curvature functions
Given positive continuous f on S^{n-1}
$f = f_K$ for a convex body $K \Longleftrightarrow \int_{S^{n-1}} u \cdot f(u) \, du = o$

Regularity theory of Monge-Ampere
Given $dS_K = f_K \, d\mathcal{H}^{n-1}$, $f_K > 0$

- f_K is C^α for $\alpha \in (0, 1] \Longleftrightarrow \partial K$ is $C_+^{2,\alpha}$
- f_K is C^k for $k \geq 1 \Longleftrightarrow \partial K$ is C_+^{k+2}

?B-M type inequality for affine surface area? Monika Ludwig, Thomas Wannerer, Andrea Colesanti, K.B.

Affine surface area

$$\Omega(K)=\int_{S^{n-1}} f_K^{\frac{n}{n+1}}\,d\mathcal{H}^{n-1}=\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\,d\mathcal{H}^{n-1}(x)$$

Theorem (Lutwak) If $n=2$ and $\alpha,\beta>0$, then

$$\Omega(\alpha K+\beta C)^{\frac{3}{2}}\geq\alpha\Omega(K)^{\frac{3}{2}}+\beta\Omega(C)^{\frac{3}{2}},$$

with equality if and only if K and C are homothetic.

(Counter)example For $n\geq 3$, there exist o-symmetric K and C

$$\Omega(K+C)^{\frac{n+1}{n(n-1)}}<\Omega(K)^{\frac{n+1}{n(n-1)}}+\Omega(C)^{\frac{n+1}{n(n-1)}}.$$

Curvature image bodies

Any convex body M in $\mathbb{R}^n$ has a unique Santalo point $s(M) \in \operatorname{int} M$ such that

$$\min_{z \in \operatorname{int} M} V((M-z)^*) = V((M-s(M))^*).$$

$\Longrightarrow$

$$\int_{S^{n-1}} u \cdot h_{M-s(M)}(u)^{-(n+1)}\, d\mathcal{H}^{n-1}(u) = o.$$

Minkowski problem $\Longrightarrow$ $\exists$ convex body CM (curvature image)

$$f_{CM}(u) = h_{M-s(M)}(u)^{-(n+1)} \text{ for } u \in S^{n-1}.$$

Theorem (Lutwak, Schneider)

If K, M convex bodies and $K \subset CM$, then

$$\Omega(K) \leq \Omega(CM),$$

with equality if and only if $K = CM$.

Affine surface area and curvature image bodies
Monika Ludwig, Thomas Wannerer, Andrea Colesanti

∂M is $C^2_+ \Longrightarrow \partial(CM)$ is C^4_+ (Monge-Ampere equations).

Theorem

$\alpha, \beta > 0$ and $N = CM$ for a convex body M with C^2_+ boundary. There exists $\delta > 0$ such that if the C^4 distance of convex bodies K and C with C^4 boundary is less than δ from N, then

$$\Omega(\alpha K + \beta C)^{\frac{n+1}{n(n-1)}} \geq \alpha \Omega(K)^{\frac{n+1}{n(n-1)}} + \beta \Omega(C)^{\frac{n+1}{n(n-1)}},$$

with equality if and only if K and C are homothetic.

?B-M type inequality for p-affine surface area? Monika Ludwig, Thomas Wannerer, Andrea Colesanti

p-Affine surface area $p \neq -n$ and $o \in \mathrm{int} K$ (Hug, Ludwig)

$$\Omega_p(K) = \int_{S^{n-1}} h_K^{\frac{n(1-p)}{n+p}} f_K^{\frac{n}{n+p}} \, d\mathcal{H}^{n-1} = \int_{S^{n-1}} (h_K^{n+1} f_K)^{\frac{-p}{n+p}} \, dV_K$$

Theorem
$n = 2$, $\frac{2}{3} \leq p \leq 1$, $\alpha, \beta > 0$, $o \in \mathrm{int}\, K$, $o \in \mathrm{int}\, C$

$$\Omega_p(\alpha K + \beta L)^{\frac{2+p}{2(2-p)}} \geq \alpha \Omega_p(K)^{\frac{2+p}{2(2-p)}} + \beta \Omega_p(C)^{\frac{2+p}{2(2-p)}}.$$

If $\frac{2}{3} \leq p < 1$, then equality holds if and only if K and C are dilates.

Remark Seems to fail completely if $p < \frac{2}{3}$ or $p > 1$

Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n}\, h_K\, dS_K$ - cone volume measure on S^{n-1} if $o \in K$ (Gromov, Milman, 1986) - also called L_0 surface area measure

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$V_K(\{u_i\}) = \frac{h_K(u_i)\mathcal{H}^{n-1}(F_i)}{n} = V(\mathrm{conv}\{o, F_i\}).$$

- $V_K(S^{n-1}) = V(K)$.

Monge-Ampere type differential equation on S^{n-1} for $h = h_K$ if μ has a density function f:

$$h \det(\nabla^2 h + h\, I) = f$$

B. Lutwak, Yang, Zhang solved in the even case

Logarithmic (L_0) Brunn-Minkowski conjecture

$\lambda \in [0,1]$, $o \in \mathrm{int}\,K, \mathrm{int}\,C$

$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle u, x\rangle \le h_K(u)^{1-\lambda} h_C(u)^\lambda \ \forall u \in S^{n-1}\}$

$\lambda K +_0 (1-\lambda)C \ \subset \ \lambda K + (1-\lambda)C$

Conjecture (Logarithmic Brunn-Minkowski conjecture)

$\lambda \in (0,1)$, K, C are o-symmetric

$$V((1-\lambda)K +_0 \lambda C) \ge V(K)^{1-\lambda} V(C)^\lambda$$

with equality iff K and C have dilated direct summands.

Conjecture (Logarithmic Minkowski conjecture)

For o-symmetric K, C, if $V(K) = V(C)$, then

$$\int_{S^{n-1}} \log h_C \, dV_K \ge \int_{S^{n-1}} \log h_K \, dV_K,$$

with equality iff K and C have dilated direct summands.

Known cases of the logarithmic B-M conjecture 1

- Interesting for any log-concave measure (like Gaussian) instead of volume
  log B-M conjecture for volume $\Longrightarrow$ log B-M conjecture for any log-concave measure (Saroglou)
- $n = 2$ for volume (Stancu + BLYZ)
- K and C are unconditional for any log-concave measure - follows directly from Prékopa-Leindler (Bollobás&Leader + Cordero-Erausquin&Fradelizi&Maurey + Saroglou on coordinatewise product)
- K and C are dilates for the Gaussian measure (Cordero-Erausquin&Fradelizi&Maurey on B-conjecture)
- Holds for the volume in $\mathbb{R}^{2n} = \mathbb{C}^n$ if K and C are complex convex bodies (Rotem)

Logarithmic B-M conjecture for almost ellipsoids

Chen, Huang, Li, Liu verified logarithmic B-M conjecture based on a result by Milman-Kolesnikov if K is close to be an ellipsoid: $\exists\varepsilon_n > 0$ such that if K, C o-symmetric with $V(K) = V(C)$ and $E \subset K \subset (1+\varepsilon_n)E$ for an ellipsoid E, then

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K,$$

with equality iff $C = K$.

Consequences of the log-B-M conjecture - Gardner-Zvavitch Conjecture

Livshyts, Marsiglietti, Nayar, Zvavitch

logarithmic B-M conjecture $\Longrightarrow$ Gardner-Zvavitch Conjecture

$$\gamma(\alpha K + (1-\alpha)C)^{\frac{1}{n}} \geq \alpha\gamma(K)^{\frac{1}{n}} + (1-\alpha)\gamma(C)^{\frac{1}{n}}$$

for o-symmetric K, C and the Gaussian measure γ on $\mathbb{R}^n$.
(γ can be replaced by any even log-concave measure)

Theorem (Kolesnikov, Livshyts)

$\int_K x\, d\gamma(x) = o$ *and* $\int_C x\, d\gamma(x) = o \Longrightarrow$

$$\gamma(\alpha K + (1-\alpha)C)^{\frac{1}{2n}} \geq \alpha\gamma(K)^{\frac{1}{2n}} + (1-\alpha)\gamma(C)^{\frac{1}{2n}}$$

L_p surface area measures

L_p surface area measures (Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K = n h_K^{-p}\, dV_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0} = nV_K$
- $S_{K,-n}$ related to $\mathrm{SL}(n)$ invariant $f_K(u)h_K(u)^{n+1}$

Theorem (Chou&Wang,Chen&Li&Zhu,B&Bianchi&Colesanti)

If $p > 0$, $p \neq 1, n$, then any finite Borel measure μ on S^{n-1} not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Remark

- Minimize $\int_{S^{n-1}} h_C^p\, d\mu$ under the condition $V(C) = 1$
- Conjectured to be unique in the even case if $0 < p < 1$

L_p Brunn-Minkowski inequality/conjecture

$p > 0$, $\lambda \in (0,1)$, $o \in \mathrm{int}\, K, \mathrm{int}\, C$

$\lambda K +_p (1-\lambda) C = \{x \in \mathbb{R}^n : \langle u, x\rangle^p \leq \lambda h_K(u)^p + (1-\lambda) h_C(u)^p \ \forall u\}$

$p \geq 1 \quad h_{\lambda K +_p (1-\lambda) C} = \left(\lambda h_K^p + (1-\lambda) h_C^p\right)^{1/p}$

L_p Brunn-Minkowski inequality/conjecture

$$V(\lambda K +_p (1-\lambda) C)^{\frac{p}{n}} \geq \lambda V(K)^{\frac{p}{n}} + (1-\lambda) V(C)^{\frac{p}{n}}$$

with equality iff K and C are dilated. Equivalent

$$V(\lambda K +_p (1-\lambda) C) \geq V(K)^{\lambda} V(C)^{1-\lambda}$$

Theorem ($p > 1$, Firey, 1962)

L_p Brunn-Minkowski inequality holds if $o \in \mathrm{int}\, K, \mathrm{int}\, C$

Conjecture ($0 < p < 1$, BLYZ, 2012)

L_p Brunn-Minkowski inequality holds if K and C are o-symmetric.

$L_0 \Longrightarrow L_p$ for $0 < p < 1$, $L_1 \Longrightarrow L_p$ for $p > 1$

The L_p Minkowski conjecture for $p_0 < p < 1$

$p_0 = 1 - \frac{c}{n^{3/2}}$

Theorem (Chen, Huang, Li, Liu)

$p_0 < p < 1$, *K, C o-symmetric*

$$V(\lambda K +_p (1-\lambda)C) \geq V(K)^{\lambda} V(C)^{1-\lambda}$$

Idea ∂K, ∂C are C^2_+ and $S_{K,p} = S_{C,p} \Longrightarrow K = C$

Step 1 (Kolesnikov, Milman)
∂M is C^2_+, $\|h_K - h_M\|_{C^2} < \varepsilon_M$ and $\|h_C - h_M\|_{C^2} < \varepsilon_M$ for $\varepsilon_M > 0$
(spectral gap for Hilbert operator)

Step 2 (Chen, Huang, Li, Liu)
Schauder estimates to get global

The Kolesnikov, Milman approach

$$D^2 h = \nabla^2 h + h\, I_{n-1} \quad \text{for } h \in C^2(S^{n-1})$$

Mixed discriminant For $h_1, \ldots, h_{n-1} \in C^2(S^{n-1})$

$$S(h_1, \ldots, h_{n-1}) = D_{n-1}(D^2 h_1, \ldots, D^2 h_{n-1})$$

Hilbert-Brunn-Minkowski operator ∂K C^2_+, $z \in C^2(S^{n-1})$

$$L_K z = \frac{S(z h_K, h_K, \ldots, h_K)}{S(h_K, \ldots, h_K)} - z$$

Theorem (Hilbert-Kolesnikov-Milman)

$L_K : C^2(S^{n-1}) \to C(S^{n-1})$ *elliptic with self-adjoint extension to* $L^2(dV_K)$

Spectral properties of $-L_K$

Trivial eigenvalues of $-L_K$

- $\lambda_0(-L_K) = 0$ (corresponding to constant functions)
- linear functions (that are odd) have eigenvalue 1 with multiplicity n

Theorem (Hilbert)

$K \in \mathcal{K}^2_+ \Longrightarrow \lambda_1(-L_K) \geq 1$

Remark: Equivalent with Brunn-Minkowski inequality

Fact $\lambda_{1,e}(-L_K) = \lambda_{n+1}(-L_K)$ for $K \in \mathcal{K}^2_{+,e}$
$\lambda_{1,e} =$first positive eigenvalue when restricted to even functions

Theorem (Kolesnikov, Milman)

$p \in [0,1)$
local L_p-Brunn-Minkowski conjecture $\Longleftrightarrow$
$\lambda_{1,e}(-L_K) \geq \frac{n-p}{n-1}$ *for* $\forall K \in \mathcal{K}^2_{+,e}$

Eli Putterman's formulation

Equivalent to L_p B-M conjecture

$p \in [0,1)$, K, L o-symmetric

$$V(K)\left((n-1)V(L[2],K[n-2]) + \frac{1-p}{n}\int_{S^{n-1}} \frac{h_L^2}{h_K}\,dS_K\right)$$

$$\leq (n-p)V(L,K[n-1])^2.$$

- If $p=1$, then we have Minkowski's second inequality
- For $p \in [0,1)$, the conjecture is stronger than Minkowski's second inequality because

$$V(K)\cdot\frac{1}{n}\int_{S^{n-1}} \frac{h_L^2}{h_K}\,dS_K \geq V(L,K[n-1])^2$$

by Hölder's inequality